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Stability of a Viscous Column of Fluid Rigidly Rotating in the Absence of Gravity

by

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ABSTRACT

Linear stability analysis techniques are used to investigate the stability of the viscous column of fluid rotating as a rigid body. The problem consisting of an infinitely long cylindrical column of fluid with constant density ρ, viscosity ν, and surface tension γ is formulated in the absence of gravity. Special cases for the inviscid problem, axisymmetric, and planar disturbances were also investigated. The inviscid problem has been studied extensively by Hocking & Micheal (1959), Hocking (1960), Gillis (1960), and Weidman, Goto, & Fridberg (1997), while the viscous problem has been studied by Hocking (1960), Gillis & Kaufman (1961), and Gillis & Sue (1962). These investigators have developed general stability criteria for both the inviscid and viscous cases including two-dimensional axisymmetric and planar disturbances, and three-dimensional spiral disturbances. The results show that rotation and viscosity, in general, have destabilizing effects. However, the general stability criteria have been shown to be independent of viscosity. Although these stability criteria have been developed, a complete analysis of stability in parameter space has yet to be completed. This report presents the formulation of the rotating viscous column of fluid as the staging for future analysis of stability in parameter space and ultimately future extension to the non-uniformly rotating problem and the viscous two-fluid problem.

INTRODUCTION

The stability of cylindrical fluid columns and jets has been widely investigated during the last century. The range of applications includes nozzle design for spray and coating devices, breakdown of vortex cores, meteorological processes, energy dissipation for hydraulic structures, and crystal growth processes in space. Initial work on this topic was pioneered by Rayleigh[9,10] in establishing stability criteria for inviscid stationary fluid columns and jets in the absence of gravity. However, interest in this field has been revived in the last 40 years. Hocking & Michael[6] found that a uniformly rotating inviscid fluid column is stable for planar disturbances of wave number n provided

$$T \geq \rho a^3 \Omega^2 [n(n+1)]^{-1},$$

where T is surface tension, ρ is the fluid density, a is the radius of the column, and Ω is the rotation rate. Hocking[5] extended these results to the limiting viscous cases of low and high Reynolds number, $Re = a^2\Omega/\nu$. Here it was shown that the rotating fluid column was stable to axisymmetric disturbances of wave number k, for both inviscid and highly viscous cases provided

$$L \geq (k^2 - 1)^{-1},$$

and planar disturbances of mode number n, provided

$$L \geq [n(n+1)]^{-1} \quad \text{inviscid},$$

$$L \geq (n^2 - 1)^{-1} \quad \text{viscous},$$

where L is the Hocking parameter (or rotational Weber number) defined as $L = T/\rho a^3 \Omega^2$. Following this work, Gillis[2] studied the viscous problem and found that stability for axisymmetric disturbances is independent of viscosity and that the stability result obtained by Hocking is valid for all Re. Gillis & Kaufman[3] extended these results by considering both azimuthal and axial disturbances to obtain the general, three-dimensional viscous criteria for stability

$$T \geq \rho a^3 \Omega^2 (a^2 k^2 + n^2 - 1)^{-1}.$$

Others including Yih[13], Gillis & Sue[4], Pedley[8], and Boudourides & Davis[1] investigated the related problem of rotating fluid columns bounded internally or externally by a solid cylinder. However, the first to extend the analysis for the inviscid problem to the complete investigation in parameter space was Weidman, *et. al.*[12]. Furthermore, they showed that exchange of stability holds for axisymmetric disturbances.

As an extension to this previous work, the viscous problem is re-formulated following the approach presented by Weidman, *et. al.*[12] for the inviscid problem and solved using techniques described by Gillis & Kaufman[3]. The problem will be further analyzed in the future and stability regions will be mapped in parameter space. The results are expected to provide an improved understanding of stability associated with a rotating viscous column of fluid. Furthermore, extension to the non-uniformly rotating problem and the two-fluid viscous problem should be possible using a similar approach.

PROBLEM FORMULATION

Problem Description

The stability of a column of constant density, viscous fluid, rotating rigidly in the absence of gravity is considered. The reference frame is taken in cylindrical coordinates (r, θ, z). In the undisturbed base state, the location of the free surface is a, the radius of the column, and the uniform rotational rate is Ω. The fluid is assumed to have constant density ρ, kinematic viscosity ν, and surface tension T.

Governing Equations

The motion of the system, in the absence of gravity, is governed by the Navier-Stokes equations that may be written in a rotating frame of reference as

$$\frac{D\mathbf{u}}{Dt} + 2(\mathbf{\Omega} \times \mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{2}\nabla(\mathbf{u} \times \mathbf{r})^2 + \nu\nabla^2\mathbf{u} \tag{1}$$

and continuity for an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u} = (u, v, w)$, $\mathbf{\Omega} = (0, 0, \Omega)$, $\mathbf{r} = (r, \theta, z)$ in cylindrical coordinates ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$). The governing equations (1) and (2) may be expanded to obtain the set of scalar equations

$$\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)u - \frac{v^2}{r} - 2v = -\frac{1}{\rho}\frac{\partial}{\partial r}(p + \frac{r^2}{2}) + \nu(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}) \tag{3}$$

$$\frac{\partial v}{\partial t} + (\mathbf{u} \cdot \nabla)v + \frac{uv}{r} + 2v = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}) \tag{4}$$

$$\frac{\partial w}{\partial t} + (\mathbf{u} \cdot \nabla)w = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w \tag{5}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

These equations may be non-dimensionalized using a for all length scales, $a\Omega$ for all velocity scales, Ω^{-1} for all time scales, and $\rho a^2\Omega^2$ for pressure.

$$\frac{\partial u}{\partial t}+(\mathbf{u}\cdot\nabla)u-\frac{v^2}{r}-2v=-\frac{\partial}{\partial r}(p+\frac{r^2}{2})+\frac{1}{Re}(\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}) \tag{7}$$

$$\frac{\partial v}{\partial t}+(\mathbf{u}\cdot\nabla)v+\frac{uv}{r}+2v=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{Re}(\nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}) \tag{8}$$

$$\frac{\partial w}{\partial t}+(\mathbf{u}\cdot\nabla)w=-\frac{\partial p}{\partial z}+\frac{1}{Re}\nabla^2 w \tag{9}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0 \tag{10}$$

where r, θ, and z are non-dimensional coordinates; u, v, and w are non-dimensional velocities; p is the non-dimensional pressure; and $Re = a^2\Omega/\nu$ is the rotational Reynolds number.

Base Flow Solution

In dimensional form, the base state in the rotating reference frame consists of u = v = w = 0, with the undisturbed free-surface located at $r = a$. Thus, the governing equations (3)-(6) reduce to a single equation for pressure

$$\frac{dp}{dr}=\rho r\Omega^2$$

subject to the boundary conditions

1.) p finite @ r = 0,
2.) p = T/a @ $r = a$.

Integrating with respect to r and applying the boundary conditions gives

$$p(r)=\frac{T}{a}+\frac{1}{2}\rho\Omega^2(r^2-a^2)$$

or, in non-dimensional form

$$p(r)=L+\frac{1}{2}(r^2-1)$$

where $L = T/\rho a^3\Omega^2$, the rotational Weber number or Hocking parameter. Thus, the base state relative to the rotational reference frame is described by

$$u=v=w=0 \tag{11}$$

$$p(r)=L+\frac{1}{2}(r^2-1);\ 0\le r\le 1. \tag{12}$$

Linearized Disturbance Equations

Equations (7)-(10) may be linearized by introducing a small parameter ε << 1, that represents a perturbation to the base state, such that

$u = \varepsilon u'$

$v = \varepsilon v'$

$w = \varepsilon w'$

$p = L + (r^2 - 1)/2 + \epsilon p'$

Substitution into the governing equations (7)-(10), subtracting the base flow solution, and neglecting higher order terms in ε gives the linearized, non-dimensional form of the governing equations where subscripts denote the respective derivatives

$$u_t - 2v = -p_r + \frac{1}{Re}\left(u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} + \frac{u_{\theta\theta}}{r^2} - \frac{2v_\theta}{r^2} + u_{zz}\right) \quad (13)$$

$$v_t + 2u = -\frac{p_\theta}{r} + \frac{1}{Re}\left(v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} + \frac{v_{\theta\theta}}{r^2} + \frac{2u_\theta}{r^2} + v_{zz}\right) \quad (14)$$

$$w_t = -p_z + \frac{1}{Re}\left(w_{rr} + \frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2} + w_{zz}\right) \quad (15)$$

$$u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_{zz} = 0 \quad (16)$$

The boundary conditions that must be satisfied include

1.) Finite velocity and pressure at $r = 0$.
2.) Continuity of normal and tangential stress at the free surface (dynamic free-surface boundary condition).
3.) Continuity of particle displacement at the free surface (kinematic free-surface boundary condition).

These boundary conditions may be non-dimensionalized and linearized accordingly to give

u, v, w, p finite	@ $r = 0$	(17a)
$p = -[\zeta + L(\zeta + \zeta_{\theta\theta} + \zeta_{zz})] - 2u_r/Re$	@ $r = 1$	(17b)
$u_\theta/r - v/r + v_r = 0$	@ $r = 1$	(17c)
$u_z + w_r = 0$	@ $r = 1$	(17d)
$u = \zeta_t$	@ $r = 1$	(17e)

where $\zeta = \zeta(\theta,z;t)$ is the position of the disturbed free surface.

Temporal Stability Analysis

The stability of the system can be investigated by temporal analysis since, in general of interest are disturbances that grow in time. So, a modal form solution may be posed such that

$$\begin{bmatrix} u \\ v \\ w \\ p \\ \zeta \end{bmatrix} = \begin{bmatrix} U(r) \\ V(r) \\ W(r) \\ P(r) \\ A \end{bmatrix} e^{i(n\theta + kz)+st} \tag{18}$$

where n is the azimuthal mode number, k is the axial wave number, A is the disturbance amplitude, and s is the growth rate defined as

$$s \equiv \sigma + i\omega, \tag{19}$$

where σ is the real growth rate and ω is the complex growth rate. Substitution of (18) into the disturbance equations and boundary conditions (13)-(17) gives

$$sU - 2V = -P_r + \frac{1}{Re}(U_{rr} + \frac{U_r}{r} - \frac{U}{r^2} - n^2\frac{U}{r^2} - 2in\frac{V}{r^2} - k^2 U) \tag{20}$$

$$sV + 2U = -\frac{in}{r}P + \frac{1}{Re}(V_{rr} + \frac{V_r}{r} - \frac{V}{r^2} - n^2\frac{V}{r^2} + 2in\frac{U}{r^2} - k^2 V) \tag{21}$$

$$sW = -ikP + \frac{1}{Re}(W_{rr} + \frac{W_r}{r} - n^2\frac{W}{r^2} - k^2 W) \tag{22}$$

$$U_r + \frac{U}{r} + in\frac{V}{r} + ikW = 0 \tag{23}$$

with boundary conditions

1.) U, V, W, P finite	@ r = 0	(24a)
2.) U = sA	@ r = 1	(24b)
3.) $P = -A[1 + L(1 - n^2 - k^2)] - 2U_r/Re$	@ r = 1	(24c)
4.) $inU - V - V_r = 0$	@ r = 1	(24d)
5.) $ikU + W_r = 0$	@ r = 1	(24e)

SOLUTION TO DISTURBANCE EQUATIONS

Following the solution approach by Gillis & Kaufman[3], introducing the transformation F = U + iV and G = U – iV gives

$$(s + 2i)F = -P_r + \frac{n}{r}P + \frac{1}{Re}\left\{F_{rr} + \frac{F_r}{r} - \left[\frac{(n+1)^2}{r^2} + k^2\right]F\right\} \tag{25}$$

$$(s - 2i)G = -\frac{in}{r}P + \frac{1}{Re}\left\{G_{rr} + \frac{G_r}{r} - \left[\frac{(n-1)^2}{r^2} + k^2\right]G\right\} \tag{26}$$

$$sW = -ikP + \frac{1}{Re}\left[W_{rr} + \frac{W_r}{r} - \frac{n^2}{r}W - k^2 W\right] \tag{27}$$

$$- 2ikW = \left[F_r + \frac{(n+1)}{r}F\right] + \left[G_r - \frac{(n-1)}{r}G\right] \qquad (28)$$

Now, (27) suggests a solution in the form of Bessel functions of order n, so assuming $W(r) = J_n(cr)$, where c is a constant to be determined, substitution into (27) and solving for P gives

$$P(r) = i\Psi\, J_n(cr) \qquad (29)$$

where

$$\Psi \equiv \frac{1}{k}\left[\frac{c^2 + k^2}{Re} + s\right] \qquad (30)$$

Then, substitution of (29) into (17) and (18)

$$F = \frac{ic\Psi}{(k\Psi + 2i)} J_{n+1}(cr) \qquad (31)$$

$$G = \frac{ic\Psi}{(k\Psi - 2i)} J_{n-1}(cr) \qquad (32)$$

Finally, these solutions must satisfy continuity by substitution of (31) and (32) into (28) to obtain

$$\Psi^3 - \frac{s}{k}\Psi^2 + \frac{4}{kRe} = 0 \qquad (33)$$

Applying the inverse transformation to (31) and (32) gives the general solution to the disturbance equations

$$U(r) = \frac{ic\Psi}{2}\left[\frac{J_{n+1}(cr)}{(k\Psi + 2i)} - \frac{J_{n-1}(cr)}{(k\Psi - 2i)}\right] = -\frac{ic\Psi}{(k^2\Psi^2 + 4)}\left[k\Psi J_n'(cr) + \frac{2in}{cr}J_n(cr)\right] \qquad (34)$$

$$V(r) = \frac{c\Psi}{2}\left[\frac{J_{n+1}(cr)}{(k\Psi + 2i)} + \frac{J_{n-1}(cr)}{(k\Psi - 2i)}\right] = \frac{c\Psi}{(k^2\Psi^2 + 4)}\left[\frac{nk\Psi}{cr}J_n(cr) + 2iJ_n'(cr)\right] \qquad (35)$$

$$W(r) = J_n(cr) \qquad (36)$$

$$P(r) = i\Psi J_n(cr) \qquad (37)$$

Thus, from (33) there exists three roots Ψ_i ($i = 1,2,3$) from which there will be three corresponding values of c_i, and hence three linearly independent solutions, the linear combinations of which in U_i, V_i, W_i, P_i will constitute the general solution to the disturbance equations. Finally, the boundary conditions will comprise a set of three homogeneous equations in the eigenvalues, λ_i such that a non-trivial solution exists provided the determinant of the coefficient matrix is zero. This gives the secular equation

$$\begin{vmatrix} P_1 + \alpha U_1 + \frac{2}{Re} U_1' & inU_1 - V + V_1' & ikU_1 + W_1' \\ P_2 + \alpha U_2 + \frac{2}{Re} U_2' & inU_2 - V_2 + V_2' & ikU_2 + W_2' \\ P_3 + \alpha U_3 + \frac{2}{Re} U_3' & inU_3 - V_3 + V_3' & ikU_3 + W_3' \end{vmatrix} = 0 \tag{38}$$

where, primes denote derivatives with respect to r and $\alpha = -[1 + L(1 - n^2 - k^2)]/s$. Equation (38) can be decomposed into

$$\mathbf{A} + \alpha \mathbf{B} = 0 \tag{39}$$

where

$$\mathbf{A} = \begin{vmatrix} P_1 + \frac{2}{Re} U_1' & inU_1 - V_1 + V_1' & ikU_1 + W_1' \\ P_2 + \frac{2}{Re} U_2' & inU_2 - V_2 + V_2' & ikU_2 + W_2' \\ P_3 + \frac{2}{Re} U_3' & inU_3 - V_3 + V_3' & ikU_3 + W_3' \end{vmatrix}$$

$$\mathbf{B} = \begin{vmatrix} U_1 & -V_1 + V_1' & W_1' \\ U_2 & -V_2 + V_2' & W_2' \\ U_3 & -V_3 + V_3' & W_3' \end{vmatrix}$$

Thus, can write the secular equation as

$$s\frac{\mathbf{A}}{\mathbf{B}} = \left[1 + L(1 - n^2 - k^2)\right] \tag{40}$$

General Stability Criteria

Gillis & Kaufman[3] suggest that **A** and **B** depend on s via the roots of (33), c_i and hence s can be determined from (40). However, equation (40) is transcendental, which makes finding all roots corresponding with stable motion virtually impossible. But, the problem may be inverted such that a scan of the region for which stable motion is likely gives a set of points for which the corresponding physical parameters may be determined to give the region of stability in parameter space. The computational results obtained by Gillis & Kaufman[3] show that the necessary and sufficient condition for stability is

$$1 + L(1 - n^2 - k^2) \geq \lim_{s \to 0}\left(s\frac{\mathbf{A}}{\mathbf{B}}\right). \tag{41}$$

Furthermore, it was also found that **A/B** remained finite as $s \to 0$ (an example of the Pellow-Southwell exchange of stability), and hence

$$\lim_{s \to 0}\left(s\frac{\mathbf{A}}{\mathbf{B}} \right) = 0 . \tag{42}$$

Thus, from (41), the motion is stable provided

$$L \geq (n^2 + k^2 - 1)^{-1}. \tag{43}$$

This is the same result obtained by Hocking[5] (1960) and Hocking & Micheal[6] (1959) in consideration of axisymmetric ($n = 0$) and planar ($k = 0$) disturbance cases separately for high and low viscosity. Those results were summarized for the rotating problem by Hocking[5] (1960) as

$L \geq (k^2 - 1)^{-1}$, axisymmetric disturbances for high viscosity;

$L \geq (n^2 - 1)^{-1}$, planar disturbances for low and high viscosity.

However, for the inviscid problem, Hocking & Micheal[6] (1959) showed stability for planar disturbances is given by

$L \geq [n(n + 1)]^{-1}$

It is interesting to note that the viscous stability criteria are independent of viscosity. Yet, it is obvious from comparison of the inviscid and viscous results for planar disturbances, that although the general stability criteria are independent of viscosity, viscosity has a destabilizing effect. So, an important question becomes how does viscosity enter into the problem? The first approach is to deduce, from the previous analysis, the eigenvalue relations obtained by Weidman, *et. al.*[12] (1997) for the inviscid problem and Hocking[5] (1960) in the limits of high and low Re and thereby show how the effect of viscosity enters into this problem.

LIMITING CASES FOR VISCOUS STABILITY

Inviscid Case

First, substitute (30) into (33) to obtain an explicit expression in c,

$$\frac{(c^2 + k^2)^3}{Re^2} + \frac{2s(c^2 + k^2)}{Re} + s^2(c^2 + k^2) + 4k^2 = 0. \tag{44}$$

Now, taking $Re \to \infty$ ($\nu \to 0$) and solving for the roots gives $c = \pm i\frac{k}{s}(s^2 + 4)$, which in turn gives for the positive root, from (30), $\Psi = \frac{k}{s}$. Thus, the solution to the disturbance equations, taking $c = i\alpha$, where $\alpha^2 = \frac{k}{s}(s^2 + 4)$, becomes

$$U(r) = -\frac{\alpha s}{2k}\left[\frac{J_{n+1}(i\alpha r)}{(s + 2i)} - \frac{J_{n-1}(i\alpha r)}{(s - 2i)} \right] \tag{45}$$

$$V(r) = \frac{i\alpha s}{2k}\left[\frac{J_{n+1}(i\alpha r)}{(s+2i)} + \frac{J_{n-1}(i\alpha r)}{(s-2i)}\right] \quad (46)$$

$$W(r) = J_n(i\alpha r) \quad (47)$$

$$P(r) = i\frac{s}{k}J_n(i\alpha r) \quad (48)$$

with the required combine kinematic and dynamic boundary condition

$$P + \frac{U}{s}\left[1 + L\left(1 - n^2 - k^2\right)\right] = 0 \qquad @\, r = 1. \quad (49)$$

Applying the boundary condition @ r = 1 gives, after some algebra, the eigenvalue relation

$$i\alpha\frac{J_n'(i\alpha)}{J_n(i\alpha)} = \left[\frac{\left(s^2+4\right)}{1 + L\left(1 - n^2 - k^2\right)} - \frac{2in}{s}\right]. \quad (50)$$

But, $i\alpha\frac{J_n'(i\alpha)}{J_n(i\alpha)} = \alpha\frac{I_n'(\alpha)}{I_n(\alpha)}$, so (51) becomes

$$\alpha\frac{I_n'(\alpha)}{I_n(\alpha)} = \left[\frac{\left(s^2+4\right)}{1 + L\left(1 - n^2 - k^2\right)} - \frac{2in}{s}\right]; \; \alpha^2 = \frac{k}{s}\left(s^2 + 4\right). \quad (51)$$

This is the same result published by Weidman, *et. al.*[12]. Obviously, in this case, viscosity is eliminated from both the disturbance equations and the boundary conditions, and hence does not appear in the eigenvalue relation. This result would seem to confirm that the general solution is correct at least in the leading behavior of large Reynolds number.

Viscous Axisymmetric ($n = 0$) Disturbances

Returning to the general solution, put $n = 0$ to obtain

$$U(r) = \frac{ic\Psi}{2}\left[\frac{J_1(cr)}{(k\Psi + 2i)} - \frac{J_{-1}(cr)}{(k\Psi - 2i)}\right] = -\frac{ick\Psi^2}{\left(k^2\Psi^2 + 4\right)}J_0(cr) \quad (52)$$

$$W(r) = J_0(cr) \quad (53)$$

$$P(r) = i\Psi J_0(cr) \quad (54)$$

where

$$\Psi^3 - \frac{s}{k}\Psi^2 + \frac{4}{kRe} = 0; \; \Psi = \frac{1}{k}\left[\frac{c^2 + k^2}{Re} + s\right] \quad (55)$$

with the required boundary conditions

$$P + \frac{U}{s}\left[1 + L\left(1 - k^2\right)\right] + \frac{2U_r}{Re} = 0 \qquad @\, r = 1 \quad (56a)$$

$$ikU + W_r = 0 \qquad @\, r = 1 \quad (56b)$$

It is important to note that there is no need to consider the disturbances in the azimuthal direction for the axisymmetric case, so V(r) is eliminated from the above solution. Then applying the boundary conditions gives the general eigenvalue relation for axisymmetric disturbances

$$c\frac{J_0(c)}{J_1(c)} = \frac{ik\beta}{ik\beta - Re} + \frac{ik\beta}{2(ik\beta - Re)}\left\{\frac{ik}{s}\left[1 + L\left(1 - k^2\right)\right] + \beta^2\right\};\ \beta^2 = c^2 + k^2 \tag{57}$$

where β and subsequently c is obtained from

$$\beta^6 + 2sRe\beta^4 + s^2Re^2\beta^2 + 4k^2Re^2 = 0. \tag{58}$$

Low Re Limit

In considering the leading behavior as $Re \to 0$ (high viscosity), equation (58) reduces to $\beta^2 = 0$, which gives

$c = \pm ik$. Furthermore, (57) reduces to

$$c\frac{J_0(c)}{J_1(c)} = 1 + \frac{1}{2}\left\{\frac{ik}{s}\left[1 + L\left(1 - k^2\right)\right]\right\};\ c^2 = -k^2, \tag{59}$$

or, making use of identities for Bessel functions

$$k\frac{I_0(k)}{I_1(k)} = 1 + \frac{1}{2}\left\{\frac{ik}{s}\left[1 + L\left(1 - k^2\right)\right]\right\} \tag{60}$$

High Re Limit

In consider the leading behavior as $Re \to \infty$ (low viscosity), (58) reduces to $\beta^2 = -4k^2/s^2$ and (57) gives

$$\frac{J_0(c)}{J_1(c)} = 0;\ c^2 = -4\frac{k^2}{s^2} - k^2.$$

This result suggests that higher order terms in *Re* should be considered to obtain the behavior of the axisymmetric case as $Re \to \infty$?

Viscous Planar (k = 0) Disturbances

Returning again to the general solution, put $k = 0$ to obtain from (30) and (33) respectively

$$\Psi = \pm\frac{2}{(sRe)^{1/2}},\ c = \pm i(sRe)^{1/2}.$$

Then, using identities for Bessel functions, the general solution reduces to

$$U(r) = -iJ_n{}'(cr) \tag{61}$$

$$V(r) = \frac{n}{cr}J_n(cr) \tag{62}$$

$$P(r) = \frac{2i}{c} J_n(cr) \tag{63}$$

where, $c^2 = -sRe$, and with the necessary boundary conditions

$$P + \frac{U}{s}\left[1 + L\left(1 - n^2\right)\right] + \frac{2U_r}{Re} = 0 \qquad @\, r = 1 \tag{64a}$$

$$ikU + W_r = 0 \qquad @\, r = 1 \tag{64b}$$

Again, in consideration of planar disturbances only, disturbances in the axial direction may be neglected. Then, applying the boundary conditions gives the general eigenvalue relation for planar disturbances

$$c\frac{J_n'(c)}{J_n(c)} = \frac{2n^2cs - c^2s + ic^2(n-2)}{2s - Re\left[1 + L\left(1 - n^2\right)\right]};\ c^2 = -sRe. \tag{65}$$

Low and High Re Limit

In considering the leading behaviors for small and large *Re* for planar disturbances the solution appears to breakdown since $c^2 = -sRe$. Thus, the solution blows up for high *Re* and reduces identically to zero for low *Re*. This behavior is not understood and may suggest and incorrect solution or it may require consideration of higher order terms in *Re* in this case as well? To investigate this, it should be possible to apply perturbation theory to obtain asymtotic approximations to c, a function of *Re*, for axisymmetric and planar cases, respectively. This is currently under investigation by solving for c in (58), assuming regular perturbation expansions in *Re* as

$c(Re) = L.B. + c_1\delta_1(Re) + c_2\delta_2(Re) + \ldots \ (Re \to 0, Re \to \infty)$,

where $\delta_1(Re)$, $\delta_2(Re)$, ... are the gage functions to be determined by considering successively higher order terms in *Re*. In this way it should be possible to obtain higher order approximations to the general solution and thus determine the effect of viscosity for all cases (i.e. axisymentric, planar, and spiral).

DISCUSSION

The important question for the stability of a rotating viscous column is what influence does viscosity have on stability. Although the general stability criteria are independent of viscosity in all cases, Hocking[5] clearly noted that for planar disturbances in the limits of high and low *Re* that the value of surface tension for the viscous problem is greater than the inviscid problem by a factor of two. That is stability is guaranteed for all *n* provided L > 1/6 for the inviscid case and L

> 1/3 for the viscous case in the limits of high and low *Re*. Thus, stability must be influenced on some level by viscosity. Both Hocking[5] and Gillis[2] offer some suggestions to this question. Hocking[5] wrote:

> "When $L < 1/(n^2 - 1)$ with viscosity neglected, the increase of capillary force at a point on the surface which is displaced outwards by the disturbance is insufficient to balance the increased centrifugal force due to the increased distance of the fluid from the axis.... For values of $L < 1/s(s+1)$ in the viscous case, the disturbance pressure is not large enough to provide this balance and the disturbance grows. One of the effects of viscosity is to alter the 180-degree phase difference between the disturbance pressure and the increase of the radius, so that the pressure no longer acts to provide a stabilizing influence at all points on the surface of the column. The instability which results from the surface tension not being strong enough to balance the centrifugal force remains, therefore, when the effect of viscosity is included, although the rate of growth of a disturbance decreases as the viscosity tends to zero."

And, Gillis[2] wrote:

> "Hence the explanation must be sought in the fact that equilibrium is not a static one , but a dynamic balance between centrifugal force and surface tension. Small perturbations of velocity or pressure lead to perturbations of the cylindrical surface, and thus to changes in surface tension force which may or may not be adequate to damp out perturbations. However, viscosity would appear to hamper this response so that a larger value of surface tension is now required to ensure stability."

Although these explanations reflect a physical description of the effect of viscosity, the mathematical description of the problem lends an additional perspective. Viscosity appears in both the disturbance equations and the boundary conditions. The role of viscosity in the disturbance equations is as always a balance between viscous forces and inertial forces such that momentum is conserved. However, it has an additional role in the free surface dynamics of this problem in which case viscosity acts to balance the dynamic free-surface conditions that ensures continuity of pressure across the free surface. Here it appears in the normal stress of the deformed or disturbed surface. Therefore, it may be expected, that under circumstances of uniform rotation that viscosity plays a negligible role in balancing inertia and it greatest effect is in the free-surface dynamics where the balance between centrifugal forces and capillary forces is most crucial as suggested by Gillis[2] and Hocking[5]. Furthermore, it may be expected for the more complicated problem of non-uniform rotation, viscosity contributes a stabilizing effect with the addition of shear in addition to the destabilizing effect inherent in the uniformly rotating problem. However, this result has yet to be systematically shown.

FUTURE WORK

Direct comparison of the results for axisymmetric and planar disturbances, in the limits of low and high *Re*, to published results by Hocking[5] and Gillis[2] is difficult since in those cases the solutions were obtained relative to a stationary reference frame. However, there should exist a transformation from a stationary reference frame to a rotational reference frame and vice versa, such that a direct comparison of results may be obtained. This transformation will be determined in the immediate future along with a complete analysis of stability in parameter space to determine preferred wave numbers or modes that produce the largest growth rates and have the greatest influence on stability. Furthermore, the case of non-uniform rotation will be investigated to obtain further insight into the effect of viscosity on stability of a rotating fluid column. Ultimately this work will be extended to the viscous two-fluid problem. Finally, a different approach to solving this problem may be possible using Squire's theorem that states:

> "To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances." – Drazin & Reid, "Hydrodynamic Stability," Cambridge University Press, 1981.

Thus, by use of Squire's transformation it may be possible to reduce the three-dimensional problem to its equivalent two-dimensional problem. If such a transformation were valid for a problem of this type, it would then be appropriate to introduce the stream function and hence obtain a form of the Orr-Sommerfeld Equation. This approach will also be investigated as follow-on to this work.

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