Analytical and Experimental Investigations of 2-D Laminar Free Jets

Dam Foundation Erosion Research Program

WATER RESOURCES MANAGEMENT.....FOR THE FUTURE

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ABSTRACT

The mechanics of two-dimensional laminar free jets of water discharged into air are investigated as a first step toward the study of jet stability. Analytic solutions are derived from the fundamental governing equations of fluid mechanics for steady, inviscid flow of an incompressible fluid in two cases: The two-dimensional laminar vertical free jet discharged from a slot, and the two-dimensional laminar horizontal free jet discharged over a weir. An experiment was designed and conducted for comparison with the analytical solution obtained for the horizontal free jet. The experimental results demonstrate good agreement with the predicted velocities for irrotational flow of an ideal fluid, neglecting contraction and aerodynamic effects. However, poor agreement was obtained in predicting jet thickness, indicating that contraction cannot be neglected in describing the shape of the jet. Together, the analytical and experimental results provide a basis for future investigations focusing on the description of free jet motion of a viscous fluid in the presence of aerodynamic effects and the analysis of jet stability.

INTRODUCTION

Background

Free jets have a wide variety of applications in the field of hydraulics. For example, reservoir outlet works often generate high pressure circular jets which are typically discharged into stilling basins downstream of a dam. In addition, high-head dams often include spillways that generate free jets which also plunge through the atmosphere into stilling basins downstream of a dam. In both cases, the jets contain considerable energy that must be dissipated in some fashion to either eliminate or at least minimize downstream erosion. Such conditions are often planned for and appropriate features are incorporated into the design. However, recent advancements in flood forecasting technology has, in many cases, increased probable maximum flood (PMF) magnitudes for existing facilities. Thus, dam owners and operators are faced with the costly prospect of retrofitting their existing facilities to meet the new PMF criteria. Such retrofits consist of increasing outlet works and spillway capacities to pass the increased PMF or raising the dam for additional storage. Neither alternative is attractive from a cost standpoint.

In response to this need for handling increased PMF criteria, a low cost alternative was identified. The idea being to allow certain existing dams to "over-top" when flood conditions produce flowrates greater than the dam is capable of either passing or storing. However, this brings into question the risk associated with such action. In particular, these conditions have significant potential for producing downstream erosion and creating structural safety problems.

In 1992, the U.S. Bureau of Reclamation (Reclamation) in cooperation with Colorado State University, initiated a research program to address the question of erosion potential at existing dams during over-topping flow conditions. Although many factors influence erosion potential, the fundamental problem that must be addressed is related to the energy associated with a free jet. Recently, Ervine, *et. al.* [41,42], Wittler, *et. al.* [146], Annandale, *et. al.* [3], and Bohrer, *et. al.* [14], have made significant progress in understanding the erosion potential of free jets. Emphasis has been placed on determining pressure fluctuations along a solid boundary produced by a plunging jet of the type that would exist in over-topping situations. However, another key aspect of the problem involves the degree of jet breakup prior to impingement. The understanding of free jet breakup phenomena for the purpose of supplementing the dam foundation erosion initiatives is the primary motivation for this research.

Problem Statement

The mechanics of steady, two-dimensional, laminar flow of water discharged into air (free jet) are investigated as a first step toward the study of jet stability.

Objectives

This research is intended to achieve an improved understanding of "over-topping" flow and the breakup phenomenon associated with jet stability. The first step is to obtain an understanding of the mechanics (*i.e.* geometric, kinematic, and dynamic aspects) of two-dimensional (2-D), laminar free jets prior to the onset of instability. An analytical representation of the inviscid problem will be developed to describe the motion prior to the onset of instability (the focus of this report). The inviscid condition will then be relaxed to obtain an analytical description of motion for the viscous fluid case. Surface tension and aerodynamics effects will then be considered and a stability analysis will be conducted. Finally, experimental results will be obtained from a physical model for comparison with the analytical results. The parameters which will be used in describing free-jet mechanics include flow depth over the crest, the initial or free stream turbulence intensity upstream of the crest, the jet velocity and thickness, the jet and ambient fluid properties, and the vertical distance from the crest to the onset of instability and subsequent breakup.

Physical Description of the Problem

The problem of the 2-D laminar free jet is closely related to the problem of over-topping flow. The investigations presented herein focus on two cases of the laminar free jet. The first case is the vertical free jet which is taken to issue into air from a sufficiently long, sharp-edged slot. Figure 1 is a schematic representation of the problem. Here the reference frame is given in Cartesian coordinates with the x-axis oriented in the vertical plane and the y-axis in the horizontal plane where

u = streamwise component of velocity (positive downward),

v = lateral component of velocity, (positive to the right),

t* = thickness along the jet,

V = resultant magnitude of axial velocity,

g = gravitational acceleration,

H = slot width.

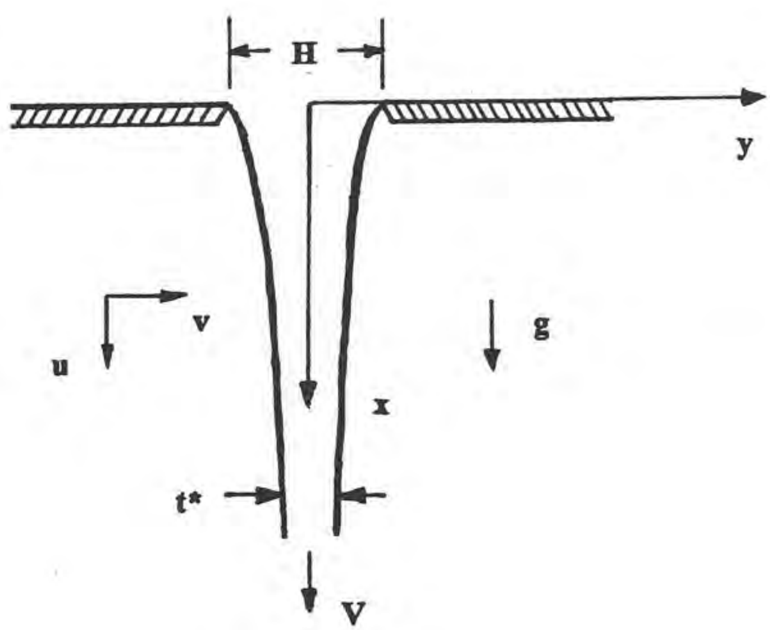

Figure 1. - Schematic of 2-D laminar flow from a slot (vertical free jet).

The second case, the horizontal free jet, is similar to the vertical free jet in that both jets have free boundaries. That is, the boundaries consist of streamlines which separate the jet (water) from the surrounding fluid (air). Figure 2 is a schematic of flow over a sharp crested weir which may also be analyzed in two dimensions assuming the weir crest to be sufficiently long. Here, the reference frame is rotated 90 degrees counter-clockwise from the vertical free jet case where

u = horizontal component of velocity (positive right),

v = vertical component of velocity, (positive to downward),

t* = lateral thickness along the jet,

V = resultant magnitude of axial velocity,

g = gravitational acceleration,

H = flow depth over the crest.

θ = angle of attack.

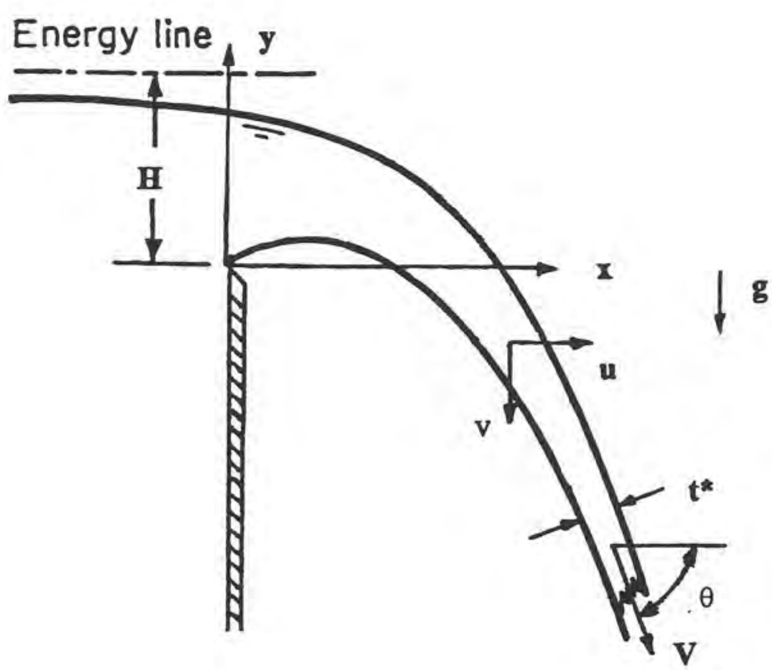

Figure 2. - Schematic of 2-D laminar flow over a sharp crested weir (horizontal free jet).

The vertical free jet was studied to provide insight into the mechanics of the horizontal free jet case. The approach was to simplify the problem to the greatest degree possible, thereby eliminating some of the geometric complexity inherent to the horizontal free jet case.

Vertical Free Jet

The vertical laminar free jet is produced by a fluid issuing from a long, narrow slot. Flow is driven through the slot under the influence of gravity. At the exit plane of the slot, the velocity profile in a viscous fluid is typically non-uniform. As the fluid springs free from the outer edges, the flow along the outer interface with the ambient fluid must accelerated to the extent that the pressure differential across the interface becomes zero. Such a phenomenon is frequently observed and is commonly referred to as contraction. Contraction is typically confined to short distances from the exit plane after which the velocity distribution across the jet becomes nearly uniform. However, the jet continues to accelerate under the influence of gravity which leads to further reduction in the jet thickness such that conservation of mass is satisfied. The problem, in this case, consists of developing an analytical description of jet motion and shape, and extending the results to the horizontal free jet case.

Horizontal Free Jet

The horizontal laminar free jet, in this context, is generated by flow passing over a "sharp-crested" weir (figure 2). The term "sharp-crested" means that the crest breadth dimension is small in relation to the depth of flow passing over it. A sheet of liquid (*i.e.* water) is produced as flow passes over the crest due to the influence of gravity. The flow is accelerated by gravity along the upper surface and the jet separates from the crest along the lower surface. Beyond the crest, the jet is considered to be free since the free streamlines separating the jet from the surrounding fluid (*i.e.* the interface) comprise the boundaries. As the jet separates from the crest, contraction takes place along the lower surface, producing a local reduction in the jet thickness. As previously mentioned for the vertical jet case, this phenomenon is a result of relaxation of the initial velocity profile which exists at the crest, to a nearly uniform velocity distribution a short distance downstream. This acceleration along the lower free streamline occurs such that the pressure along the interface becomes equal to the ambient pressure. At this point (*i.e.* complete contraction), the velocity distribution is essentially uniform. However, the jet continues to accelerate under the influence of gravity and is deflected downward similar to the motion of a projectile in a uniform gravitational field. During this acceleration, the jet thickness continues to decrease as a consequence of continuity. At some point the jet becomes unstable and disturbances grow rapidly to a point at which the jet begins to breakup. The breakup phenomenon itself is little understood and is the ultimate focus of this research. However, before addressing the stability problem, a description of the 2-D laminar free jet motion prior to the onset of instability is required.

ANALYTICAL APPROACH

As previously mentioned, the inviscid vertical free jet is investigated first and the results are extended to the inviscid horizontal free jet.

Vertical Free Jet

The fundamental equations of fluid mechanics are used to describe the motion of the free jet. For two-dimensional steady flow of an incompressible viscous fluid, the general set of governing equations, expressed in Cartesian coordinates as given in figure 1 are

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{1}$$

$$u\partial u/\partial x + v\partial u/\partial y = -(1/\rho)\partial p/\partial x + \nu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) + g, \tag{2}$$

$$u\partial v/\partial x + v\partial v/\partial y = -(1/\rho)\partial p/\partial y + \nu(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2). \tag{3}$$

Eqn. (1) is the continuity equation that requires mass be conserved. Eqns. (2) and (3) are the momentum equations in the x and y directions, respectively and are commonly referred to as the Navier-Stokes equations. The temporal derivatives $\partial u/\partial t$ and $\partial v/\partial t$ have been eliminated from the equations as a consequence of steady flow. The third fundamental equation, the energy equation, is not required for this problem since it is mathematically uncoupled from the equations of motion for flow of an incompressible fluid at constant temperature. Thus, a mathematical description of 2-D free jet motion (*i.e.* velocity and shape) can be obtained from Eqns. (1)-(3). However, difficulty arises in obtaining an exact solution to Eqns. (2)-(3). Thus, certain simplifying assumptions are required.

Simplifying Assumptions

It should first be pointed out that aerodynamic effects (*i.e.* the interaction between the jet and the surrounding air) are neglected for this analysis. This simplification may not be applicable for the future stability analysis where such effects (*i.e.* interfacial shear) may have a significant influence on jet stability. Due to the contraction phenomenon immediately downstream of the slot, difficulty arises in applying the Eqns. (1)-(3) to describe jet motion in this region. There is no justification here for further simplification, since relatively large spatial gradients of pressure and velocity exist over the reach for which contraction occurs. Contraction has previously been described analytically using free streamline theory (Lamb[13] and Milne-Thomson[16]) and other techniques discussed in the literature review. Free streamline theory is rather complex and will not be dealt with in this analysis. Instead, contraction effects will be neglected and a uniform velocity distribution will be assumed to exist at the exit plane. As a future extension of this research, free streamline theory or other appropriate mathematical techniques will be used to describe jet motion in the contraction region. However, as a first and simple approach to the analysis, viscous effects are neglected, pressure gradients are assumed to be zero, and the flow is assumed to be irrotational.

Euler's Equations

Neglecting viscous effects reduces Eq'ns. (2)-(3) to the well known Euler's equations of motion for an ideal (*i.e.* incompressible and inviscid) fluid. These equations along with continuity become

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{4}$$

$$u\partial u/\partial x + v\partial u/\partial y = -(1/\rho)\partial p/\partial x + g, \tag{5}$$

$$u\partial v/\partial x + v\partial v/\partial y = -(1/\rho)\partial p/\partial y. \tag{6}$$

Next, assuming the flow to be irrotational implies that vorticity is zero which is represented for 2-D flow as

$$\partial v/\partial x = \partial u/\partial y. \tag{7}$$

Substitution of Eqn. (7) into Eqns. (5) and (6) gives

$$u\partial u/\partial x + v\partial v/\partial x = -(1/\rho)\partial p/\partial x + g, \tag{8}$$

$$u\partial u/\partial y + v\partial v/\partial y = -(1/\rho)\partial p/\partial y. \tag{9}$$

Eq'ns. (8) and (9) can be integrated to obtain

$$u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial x)\, dx - gx = C_1, \tag{10}$$

$$u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial y)\, dy = C_2. \tag{11}$$

For steady, irrotational flow $C_1 = C_2$ for the entire flow field, hence Eqns. (10) and (11) can then be combined to obtain the well know Bernoulli equation written as

$$u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial x)\, dx - gx = u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial y)\, dy. \tag{12}$$

This result is valid for the entire flow field assuming steady, irrotational flow and indicates that the total energy is conserved. Applying this result in the absence of pressure gradients to the problem at hand gives for conditions at the exit plane defined by $u(0) = u_o$, $v(0) = 0$

$$u^2 + v^2 - 2gx = u_0^2 \tag{13}$$

But, $u^2 + v^2 = V^2$, so the resultant velocity along the jet is

$$(u^2 + v^2)^{1/2} = V = (u_0^2 + 2gx)^{1/2}. \tag{14}$$

The magnitude of velocity and the respective components are described analytically by Eqn. (14), assuming contraction at the exit plane can be neglected. It can be shown that Eqn. (14) is the same result that is obtained from the parametric equations for a free-falling body under a uniform gravitational field, neglecting drag. If contraction cannot be neglected, then the problem becomes one of determining the jet geometry and velocity at the section where the velocity distribution is uniform (*i.e.* where contraction is complete). This result then becomes the necessary boundary condition for Eqn. (14) in describing the subsequent motion of the jet between the point of complete contraction and the onset of instability.

Continuity

Having an expression for the resultant magnitude of velocity the jet thickness can also be described analytically. The expression is obtained by defining the two-dimensional streamline geometry that satisfies continuity

$$t_o^* V_o = t^* V, \tag{15}$$

where, t_o^* and t^* are the distances between the free streamlines (*i.e.* the lateral jet thicknesses) at any two cross sections along the jet, and V_o and V are the respective velocities. Solving for t^* and substituting Eqn. (14) for V gives

$$t^* = (t_o^* V_o)/(u_o^2 + 2gx)^{1/2}. \tag{16}$$

Neglecting contraction at the exit plane, letting $t_o^* = H$, and $V_o = u_o$, the non-dimensional jet thickness at any distance below the exit plane is determined as

$$t^*/H = u_o/(u_o^2 + 2gx)^{1/2}. \tag{17}$$

Together, Eqns. (14) and (17) describe the geometric and kinematic aspects of motion for the two-dimensional, laminar vertical free jet of an ideal fluid in the absence of aerodynamic effects. Figures 3 and 4 are graphical representations of jet velocity and non-dimensional thickness for various exit velocities u_0 as a function of distance x below the exit plane.

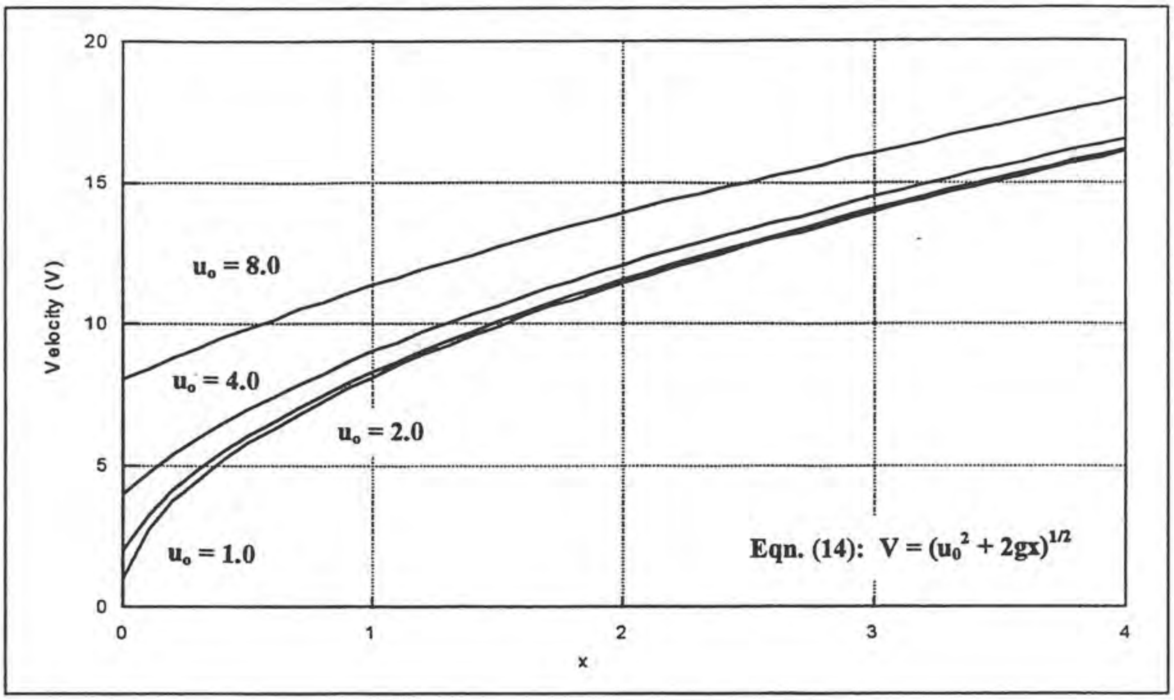

Figure 3. - Jet velocities as a function of distance from the exit plane. Each curve represents increasing values of u_0.

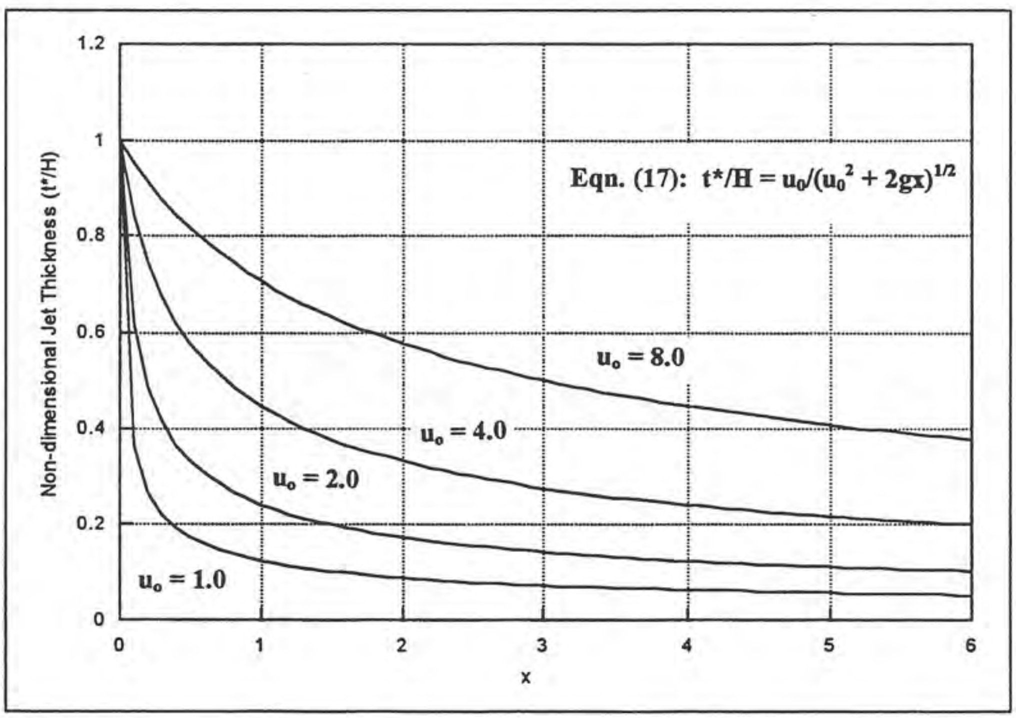

Figure 4. - Jet shapes represented by non-dimensional jet thickness as a function of distance from the exit plane. Increasing values of u_0 represent decreasing rates of reduction in thickness.

Horizontal Free Jet

The analytical results for the vertical free jet are easily extended to the horizontal free jet. Modifying the frame of reference to that given in figure 2, the Bernoulli equation becomes

$$u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial x)\, dx = u^2/2 + v^2/2 + 1/\rho \int (\partial p/\partial y)\, dy - gy. \tag{18}$$

Assuming no pressure gradients and defining conditions at the crest to be $u(0) = u_0$ and $v(0) = 0$ gives

$$(u^2 + v^2)^{1/2} = V = (u_0{}^2 - 2gy)^{1/2}\,. \tag{19}$$

Again, this is the same result that is obtained in combining the parametric equations for describing particle trajectory motion in the absence of aerodynamic affects.

Continuity

As before, the two-dimensional streamline geometry that satisfies continuity is defined by

$$t_o{}^*V_o = t^*V, \tag{20}$$

where $t_o{}^*$ and t^* are the distances between the free streamlines at any two cross sections along the jet, and V_o and V are the respective velocities. Solving for t^* and substituting Eqn. (19) for V gives

$$t^* = (t_o{}^* V_o)/(u_o{}^2 - 2gy)^{1/2}. \tag{21}$$

Again, neglecting contraction just downstream of the crest, letting $t_o{}^* = H$, and $V_o = u_o$ the non-dimensional jet thickness at any distance below the crest is given as

$$t^*/H = u_o/(u_o{}^2 - 2gy)^{1/2}. \tag{22}$$

Eqns. (19) and (22) describe the geometric and kinematic aspects of motion for the two-dimensional, laminar horizontal free jet of an ideal fluid in the absence of aerodynamic effects.

As an extension of the preceding analysis, horizontal and vertical components of velocity obtained from Eqn. (19) may be integrated to give the parametric equations for position as a function of time. The components of velocity are

$$u = V_x = V_0 \cos\theta, \tag{23}$$

$$v = V_y = V_0 \sin\theta - gt. \tag{24}$$

Where, θ is the jet angle of attack between the unit tangent vector and the horizontal at each point along the jet (figure 2). Since, $u = dx/dt$ and $v = dy/dt$, integration of Eqns. (23) and (24) with respect to time gives,

$$x(t) = V_0 t\cos\theta + C_1, \tag{25}$$

$$y(t) = V_0 t\sin\theta - gt^2/2 + C_2 \tag{26}$$

The constants of integration are obtained by applying the initial conditions $x = y = 0$, at $t = 0$. Thus, $C_1 = C_2 = 0$. Combining Eqns. (25) and (26) by eliminating t gives,

$$y = x\tan\theta - gx^2/(2V_0^2\cos^2\theta) \tag{27}$$

Eqn. (27) can be applied to any streamline to obtain the direction of the unit tangent vector at all points along the jet.

Viscous Effects

The importance of viscous effects in this analysis are confined to the contraction region of the jet. However, in general, viscous effects are also important in considering jet stability. Having established a description of motion for the jet in the absence of contraction, the next step is to apply other analytical techniques to describe contraction and match the solution with that obtained above. Although contraction is difficult to handle analytically and is further complicated by consideration of viscosity, this problem has been studied extensively in the past and the various analytical techniques used to describe contraction will be reviewed in the very near future as the next step in this research.

LITERATURE REVIEW

Initially the literature search focused on aspects of stability to ascertain whether the ultimate objective of this research, a stability analysis has been previously studied in detail. Although a great deal of literature exists on the stability of free jets in air, much of it is confined to the axisymmetric free jet case. Notably, one of the first to study stability of axisymmetric free jets was Rayleigh [103,104]. However, his analyses were confined to the linearized stability neglecting aerodynamic effects. Since that time, capillary stability has been studied by Alterman [1], Bechtel, *et. al.*[5], Bogy [8], Carlson [17], Chaudhary, *et. al.* [19,20,21], Dombrowski, *et. al.* [27,28], Donally, *et. al.* [29], Eggers, *et. al.* [32,33], Goedde and Yuen [49], Goren, *et. al.* [50], Grant, *et. al.* [51], Keller [62], Kistler, *et. al.* [69], LaFrance [72,73], Mansour, *et. al.* [84], Rutland and Jameson [109], Tjahjadi [120], and Yuen [133] for a variety of axisymmetric free jet conditions. Much of this work has been successful in analytically describing jet breakup in the presence of artificial disturbances and aerodynamic effects. However, in many cases gravitational forces are neglected. For the horizontal free jet, gravitational forces are predominant. Thus, precluding gravity from the analysis, in this case, is not justified.

Having completed the preliminary literature search and review, it was apparent that the first step in studying the horizontally discharged free jet was to describe the motion prior to the onset of instability. Thus, a literature search was conducted to ascertain the extent of previous work (theoretical and experimental) on the study of two-dimensional laminar free jets (vertical and horizontal). Concurrent with the literature search was an independent development of the analytical descriptions of jet motion for the two cases previously identified using the fundamental equations of fluid mechanics. This approach allowed for familiarization with the problem at hand and made the literature review a more efficient process.

As with the study of capillary instability, a great deal of literature exists on the study of two-dimensional free jets in air. Most notably are the theoretical studies conducted by Bogy [10], Brown [14], Brun, *et. al.* [15], Campbell [16], Clark [24,25], Duda [30], Faiz [36], Gavis, *et. al.* [40, 41], Geer [42,43,44,45,46], Huang [57], Kaye, *et. al.* [60], Kandaswami, *et. al.* [61], Keller, *et. al.* [63,64,65,67], Lance, *et. al.* [75], Lee [78], Lienhard [81, 82], Nickell, *et. al.* [93], Page [97], Pai [98], Richardson [105], Rupe [108], Scrivien, *et. al.* [112], Sussman, *et. al.* [115], Taylor [118], Ting, *et. al.* [119], Tuck [121], Tyler, *et. al.* [122], and Wang [129]. However, in most cases, contraction is not considered.

Regarding two-dimensional or plane liquids sheets, Dombrowski, *et. al.* [27,28], Frazer, *et. al.* [38,39], Hagerty [54], Huang, *et. al.* [57,58], Lance, *et. al.* [75], and York, *et. al.* [132] have investigated aspects of stability.

The effects of contraction have been considered in a number of the sources found to date including work by Campbell [16], Faiz [36], Gavis, *et. al.* [40,41], Goren, *et. al.* [50], Kandaswami [61], Lee [78], Leinhard [81,82], Middleman, *et. al.* [88], Nickell, *et al.* [93], Oliver [96], Richardson [105], Rupe, *et. al.* [108], Tyler, *et. al.* [122].

Only two sources have been found, to date, that apply numerical techniques to the study of two-dimensional laminar free jets which include work by Faiz [36], and Nickell, *et. al.* [93]. Fiaz dealt with the steady axisymmetric and two-dimensional vertical free jet and solved the problem (*i.e.* jet velocities and shape) considering aerodynamics effects. Nickell, *et al.* also solve the steady axisymmetric and two dimensional problems numerically in the absence of gravity and with full consideration given to contraction and expansion effects at the exit plane. The former being the case at low Reynolds numbers and the later at high Reynolds numbers. In both cases, the inherent difficulty with the problem involved the *a priori* unknown free boundary.

Finally, a thorough review of literature related to free jets was conducted in 1956 by Krzywoblocki [71], but has yet to be acquired. Much of the literature cited in the bibliography has yet to be reviewed. Presently, the review has been limited to studies of two-dimensional laminar free jets in air because of the importance in understanding the related motion prior to the study of stability. A complete review of the cited literature will be conducted in the next six months and will undoubtedly turn up additional sources for review as this is expected to be a continuous process throughout the course of this research.

EXPERIMENTAL APPROACH

Dimensional Analysis

A dimensional analysis was conducted for the purpose of experimental design. The Buckingham π theorem was used to establish non-dimensional parameters from the set of physical quantities thought to be involved in the mechanics of the two-dimensional horizontal laminar free jet problem. The relationship between physical quantities involved in describing jet motion can be written mathematically as

$$f(u, v, t^*, H, g, \rho_a, \rho_w, \nu_a, \nu_w, \sigma, Tu) = 0, \tag{28}$$

where,

u = horizontal component of velocity $[LT^{-1}]$;

v = vertical component of velocity $[LT^{-1}]$;

t^* = jet thickness $[L]$;

H = flow depth over the crest $[L]$;

g = gravitational acceleration $[LT^{-2}]$;

ρ_a = density of air $[ML^{-3}]$;

ρ_w = density of water $[ML^{-3}]$;

ν_a = kinematic viscosity of air $[L^2T^{-1}]$;

ν_w = kinematic viscosity of water $[L^2T^{-1}]$;

σ = surface tension at water-air interface, $[MT^{-2}]$;

Tu' = turbulence intensity $[LT^{-1}]$;

The variables H, g, and ρ_w were selected as repeating for this analysis, H being the primary independent variable for the experiment with g and ρ_w being assumed constant. The following π

parameters were obtained:

$\pi_1 = u/(gH)^{1/2}$ [Fr];

$\pi_2 = v/(gH)^{1/2}$ [Fr];

$\pi_3 = t^*/H$;

$\pi_4 = \nu_w/(gH^3)^{1/2}$ $[1/Re]^2$;

$\pi_5 = \nu_a/(gH^3)^{1/2}$ $[1/Re]^2$;

$\pi_6 = \rho_a/\rho_w$;

$\pi_7 = \sigma/(\rho_w g H^2)$ [1/Wb];

$\pi_8 = Tu'/(gH)^{1/2}$ [Fr].

Together, these π parameters provide a basis for analysis of the experimental data since any one π parameter can be written as a function of all the other π parameters. Given the objectives of this project, which place emphasis on the case where initial disturbances are minimized, the results can be simplified. For the experimental phase of this project turbulence intensity, Tu', will be minimized. This eliminates the corresponding π_8 parameter from the analysis. Thus, the primary dependent variables of the experiment consist of u, v, and t*. The primary independent variable was selected as H and will be varied during the experiments. The remaining physical quantities: ρ_a, ρ_w, ν_a, ν_w, σ, and g will be assumed to remain constant during the experiments.

Physical Model Description

The physical model was designed by R. Wittler, Ph.D. and myself, and constructed at Reclamation's Water Resources Research Laboratory (Hydraulics Laboratory) in Denver, Colorado. The model consists of a 2.0-m square headbox that is 3.0-m deep with a 1.0-m long "sharp-crested" weir installed along the side. The headbox was raised above the laboratory floor so that the crest elevation is located approximately 5.0 m above the laboratory floor. Figure 5 is a photograph of the physical model. Flow is supplied to the head box via an 8.0-in diameter delivery pipe by means of a "high-head," vertical turbine pump. The delivery pipe supplies flow to the upstream side of a baffle that was installed to settle the flow (*i.e.* minimize free stream turbulence).

The total head was raised above the crest elevation and a quasi-two dimensional, horizontally discharged free jet was produced.

Figure 5. - Photograph of physical model as constructed at Reclamation's WRRL.

The crest length was specified to be 1.0 m to obtain jet velocity and thickness measurements sufficiently far from edge effects produced by contraction along the sides of the jet. Aspect ratios of jet thickness to width at the crest greater than 20:1 were investigated.

Preliminary testing revealed that the maximum total head, H, in the physical model was limited to approximately 5.5 cm above the crest. This was due to the inadequacy of the baffling to settle higher discharges and correspondingly greater flow depths. As previously indicated, the emphasis for these investigations was limited to the laminar case in which initial disturbances were minimized to the degree possible. Thus, testing was conducted over a range of H values from 1.50 cm to 5.50 cm. In the future, the baffling will be improved to accommodate higher values of H for testing.

The total head depth in the head box was measured using a hook gage with Vernier scale graduated to 0.1 mm. The hook gage is used to obtain an accurate measurement of water surface elevation above some known datum (*i.e.* the crest in this case). The hook gage was installed in a stilling well which was connected to the head box. The stilling well is required to minimize measurement difficulties due to surface waves, should they exist. This setup was located along the right side of the head box (figure 5).

Initially, acquisition of velocity data was attempted using a hot-film anemometer in the hopes that any initial disturbances, that may exist in the form of turbulent fluctuations, could be evaluated. However, difficulties in operation of the available instrument precluded such measurements. Instead, it was necessary to obtain velocity measurements using a back-flushing pitot tube. The back-flushing pitot tube measures local static and dynamic pressure simultaneously which could then be related to velocity head through the relation:

$$P_{dynamic} = P_{meas} - P_{static} = V^2/2g \tag{29}$$

The back-flushing feature was required since the jet thickness at some of the measurement locations was on the order of the pitot tube dynamic pressure port diameter of 0.10 cm, and had the potential to draw air into the pressure lines as the probe was being positioned. Furthermore, at lower values of $H < 1.5$ cm, considerable "wobble" in the jet was observed indicating the jet to be close to the stability limit and precluding accurate velocity measurements. The static and dynamic pressure lines of the pitot tube were connected to a differential pressure transducer. The pressure transducer was then connected to a digital display (LED) and transmitter. The transmitter outputs an analog signal of 0-5 volts that corresponds with full scale of differential pressure, 0-10 psi (0-

23.1 ft H_2O). The transmitter was patched into a data logger for communication with a laptop PC. Data were acquired at a sampling rate of 25 Hz. Measurements were acquired at approximately 0.3-m elevation intervals below the crest. The probe elevation at each measurement location was determined by closing the static pressure valve and manually obstructing the dynamic pressure port to obtain a measurement of total head below the water surface elevation in the head box. This value could then be related, hydrostatically, to the depth of flow over the crest and hence to the probe distance below the crest. Figure 6 is a photograph of the back-flushing pitot tube. Jet thickness measurements were acquired using a point gage in combination with a Vernier scale graduated to 0.1 mm. Thickness measurement locations corresponded with jet velocity measurements. Figure 7 is a photograph of the point gage setup.

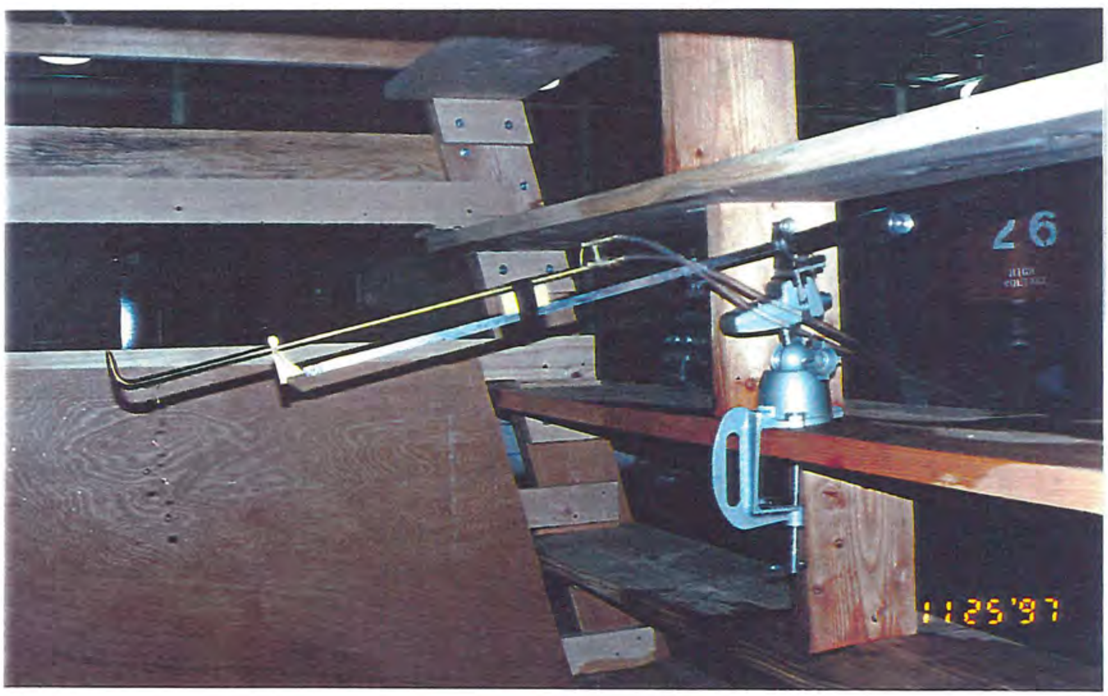

Figure 6. - Photograph of back-flushing pitot tube used to acquire jet velocity measurements.

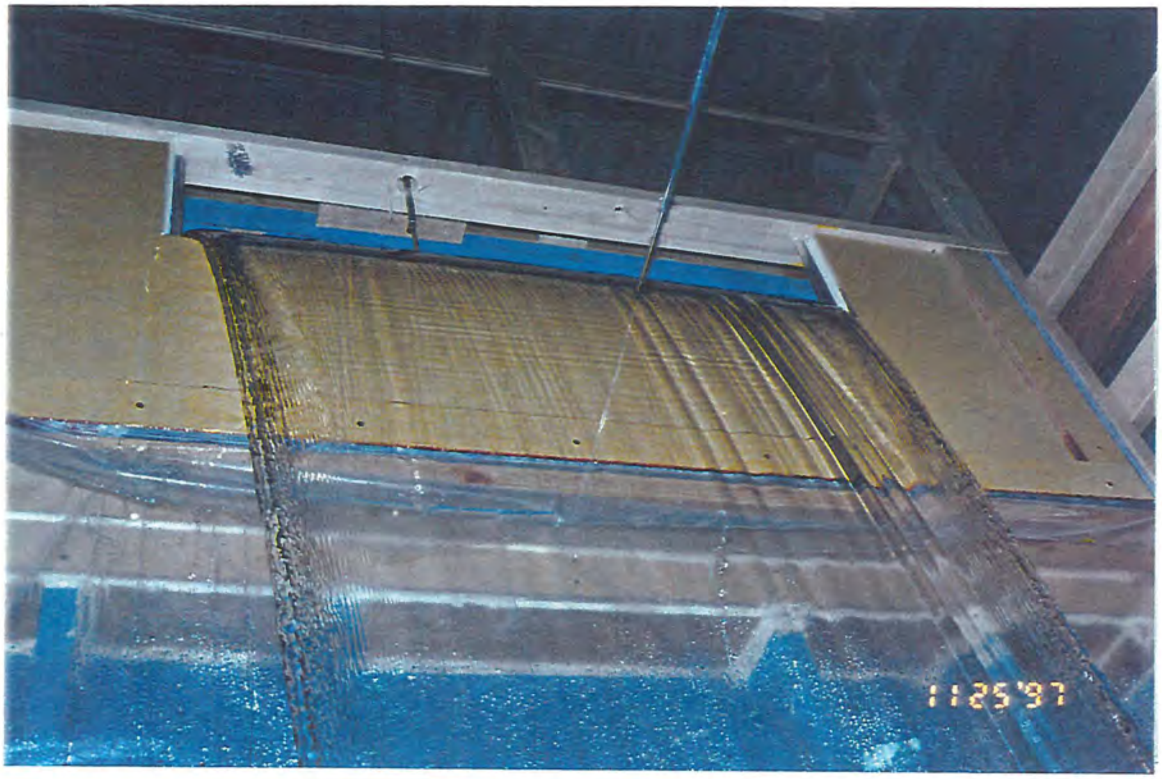

Figure 7. - Photograph of point gage used to acquire jet thickness measurements. Note the side contraction or edge effects inherent to the physical model.

RESULTS

The analytical and experimental results for jet velocity and geometry have been compared. The results from two completed test runs are presented. Run #1 was the first completed and represents questionable results. Run #2 consists of improved experimental methods and consequently improved results. Both sets of velocity results are presented for the purpose of discussing some of the difficulties encountered during the experiments. The velocity results are presented as velocity verses vertical distance below the crest. Similarly, jet thickness results are presented as thickness verses vertical distance below the crest and non-dimensional jet thickness t*/H as a function of vertical distance below the crest.

Velocity Results

The analytical and experimental results describing jet velocity as a function of vertical distance below the crest are shown as Figures 8 and 9. The data represent different values of total head depth over the crest, H, ranging from approximately 1.50 cm to 5.50 cm. As previously indicated 5.50 cm was the upper limit above which initial disturbances become significant. The analytical results are included with the experimental data for comparison. Eqn. (19) is plotted for the upper and lower H values of the range tested. The initial velocity, u_0 is defined as the average velocity above the crest and is obtained from the correlation for unit discharge over a sharp crested weir, Rouse [106]:

$$q = (0.667)C_d(2gH^3)^{1/2}, \tag{30}$$

Where, C_d is the discharge coefficient obtained experimentally, g is gravitational acceleration, and H is the flow depth above the crest. The discharge coefficient has been found experimentally to be

$$C_d = 0.611 + 0.075(H/L), \tag{31}$$

where L is the crest length. Thus, the average velocity is obtained as

$$V_{avg} = u_0 = q/H. \tag{32}$$

Eqns. (30)-(32) establish a relationship between the independent variable H (*i.e.* control variable for this experiment) and the initial velocity used in the analytical expression, Eqn. (19).

For Run #1, the solid line represents the predicted velocity as a function of distance below the crest for H = 4.32 cm while the dashed line is predicted velocity for H = 2.18 cm. Together these two lines represent the upper an lower limits of H values tested. The results show considerable scatter in the data believed to be a consequence of the data acquisition methods used during Run #1.

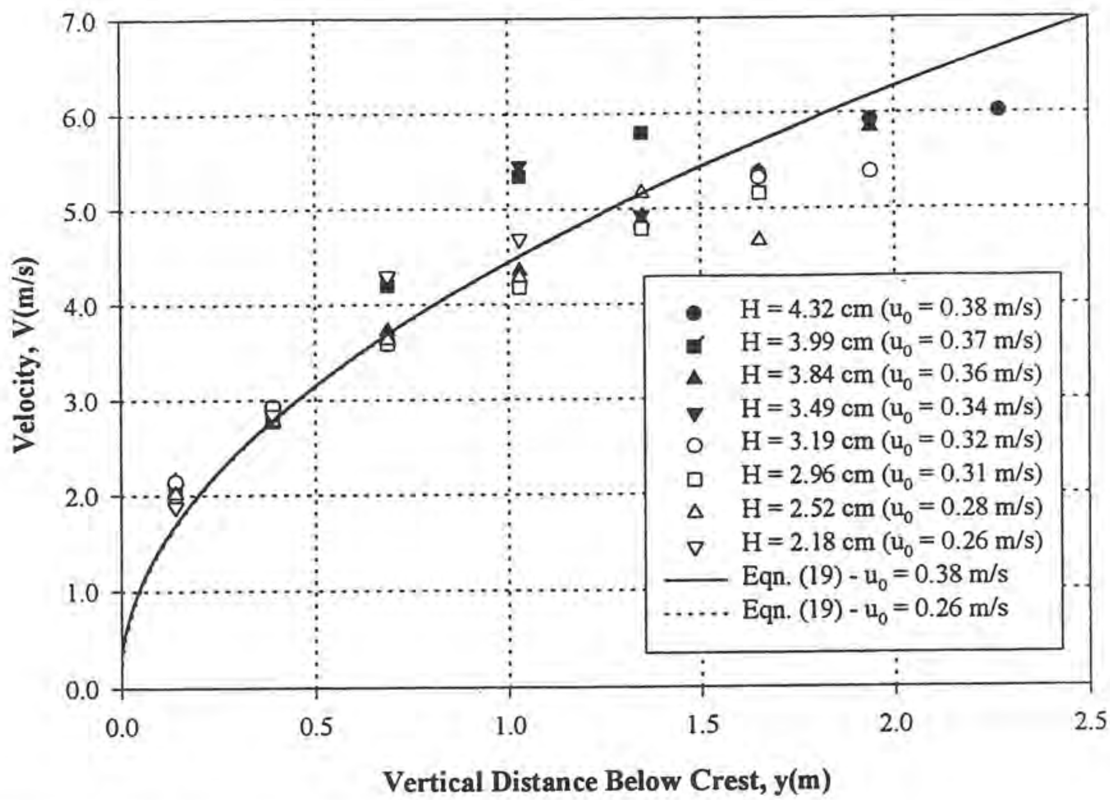

Figure 8. - Run #1 results: Velocity verses vertical distance below the crest.

During Run #1, data were acquired at each measurement location (*i.e.* vertical distance below the crest) for each fixed value of H. This required moving the pitot tube mounting fixture several times during each test. Such a method creates difficulty in acquiring velocities at precisely the same location for each value of H tested. After obtaining these results, the data acquisition method was altered. The modified procedure consisted of measuring velocity for a range of H values at each measurement location and produced a considerable improvement in the results, as expected. In particular, the previously seen scatter in the data was virtually eliminated. Figure 9 shows the results of Run #2. Again, the solid and dashed lines represent the predicted velocities for the upper and lower H values of the range tested. A comparison of the predicted results, Eqn. (19), and the experimental observations shows good agreement. However, the most important result is that the observed velocity at each elevation below the crest is independent of H for values of vertical distance below the crest greater than 0.12 m.

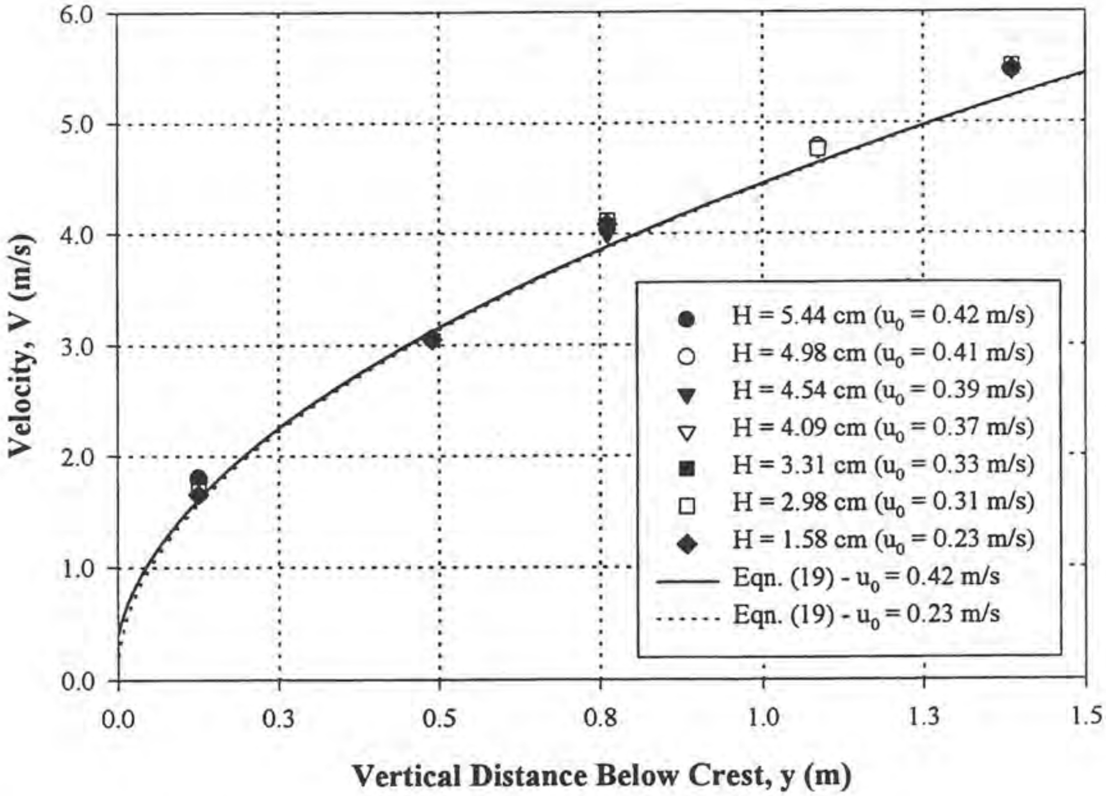

Figure 9. - Run #2 results: Velocity verses vertical distance below the crest.

This result is as expected since each of the increasing H values tested represents a small increases in initial velocity. Thus, for the range of initial velocities tested being small, gravitational effects are increasingly predominant at increasing distances below the crest. At distances less than 0.12 m, contraction effects predominantly influence jet thickness.

Jet Thickness Results

The analytical and experimental results describing jet thickness as a function of vertical distance below the crest, for Run #2, are plotted in figure 10. In addition, the results are presented as figure 11 for non-dimensional jet thickness, t*/H as a function of vertical distance below the crest. As with the velocity results, the data are plotted for different values of H. A comparison of experimental observations with the results predicted by Eqn. (22) shows limited agreement. For lower values of H, Eqn. (22) appears to closely predict jet thickness as a function of vertical distance below the crest. However, for larger H values, Eqn. (22) over-predicts jet thickness. This is most likely due to contraction effects at the crest. As previously mentioned, contraction rapidly decreases local jet thickness which is not accounted for in the analytical development of Eqn. (22). Hence, improvements are required and may be achieved by application of free streamline theory or other analytical techniques to be applied as future work for this research project. Subsequently, describing the contraction phenomenon will allow for a solution in terms of jet thickness and velocity at the point when contraction is complete. This solution can then be matched with Eqns. (19) and (22) which appear to adequately describe jet motion downstream of the contraction region, hence giving a complete description of jet motion from the crest to the point of instability in the absence of aerodynamic effects. Thus, a basis will exist for which a stability analysis may be conducted.

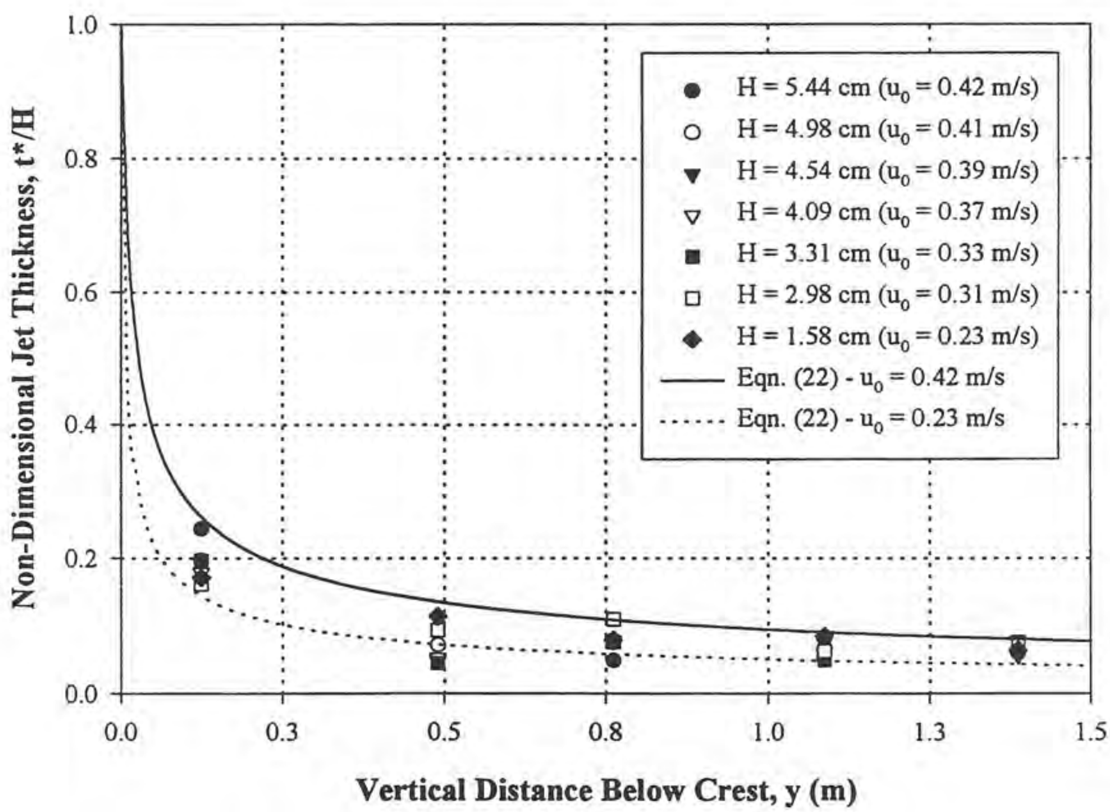

Figure 10. - Run #2 results: Non-dimensional jet thickness verses vertical distance below the crest.

Non-Dimensional Relationships

Figure 11 represents non-dimensional jet thickness, t*/H as a function of Froude number, Fr. In this case Fr is defined as obtained from the dimensional analysis,

$$Fr = V/(gH)^{1/2}. \tag{33}$$

Where, V is the velocity measured along the jet, and H is the flow depth over the crest. The Froude number represents the relative relationship between inertial forces and gravitational forces. Thus, the results (figure 11) indicate that gravitational forces become decreasingly predominant at increasing distances below the crest. This result is intuitive since as the flow accelerates, inertial forces become increasingly predominant. Thus, the divergence between relative magnitudes of inertial and gravitational forces indicating for Fr > 4.0, the non-dimensional jet thickness is less sensitive to increases in Froude number.

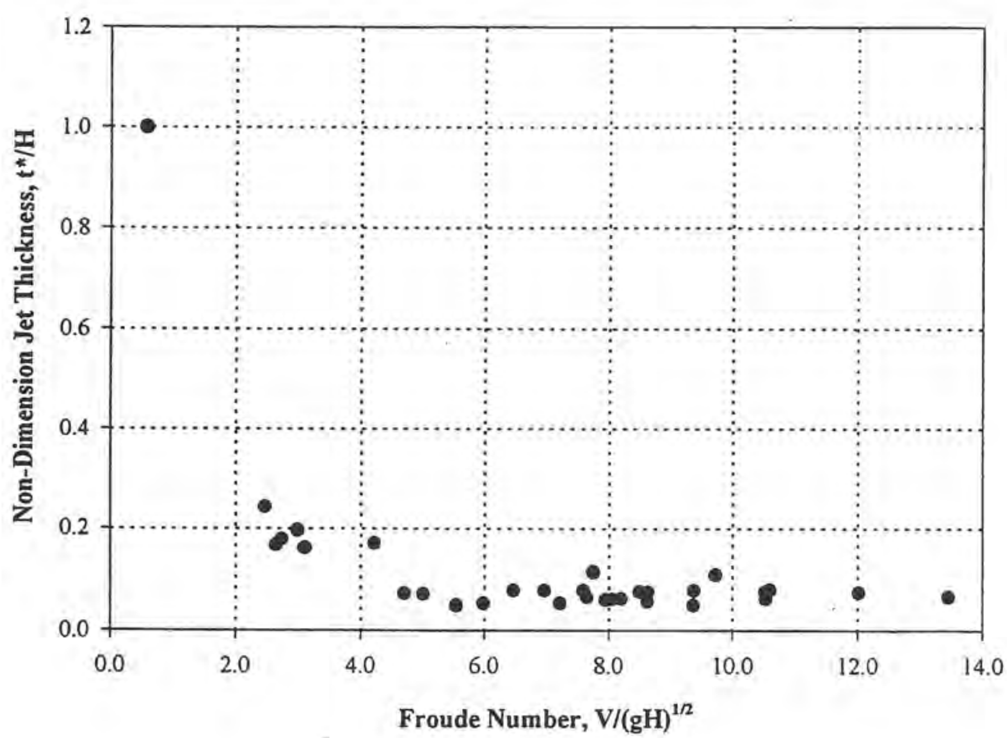

Figure 11. - Non-dimensional jet thickness verses Fr along the jet.

Figure 11 shows a distinct relationship between non-dimensional jet thickness and Froude number, from which a relationship between jet thickness and velocity may be obtained for other values of flow depth over the crest, H. In the future these results will be used in conjuction with other non-dimensional parameter relationships to obtain a complete relationship for free jet velocity and thickness as functions of flow depth over the crest, initial or free stream turbulence, vertical distance below the crest, and respective fluid properties. In that way correlations may be developed for application of results to much larger scales consistent with typical prototype field conditions.

Qualitative Results: Nature of Instability

Although the nature of instability for the two-dimensional laminar free jet was not quantitatively described, an interesting qualitative description was obtained. Figures 12 and 13 show the qualitative nature of instability. Figure 12 is a side view photograph of the jet produced during the experiments and illustrates the growth of disturbance during the onset of instability. Such phenomena are exhibited frequently in nature (*i.e.* transition from laminar to turbulent flow). Figure 13 is an oblique view of the jet and illustrates the degree of structure associated with the flow. In particular, it appears that two modes of instability may exist: The lateral growth of disturbance along the jet; and a "ribbon-like" breakup across the jet. In any case, figures 12 and 13 provide an interesting perspective on the instability problem to be investigated in the future.

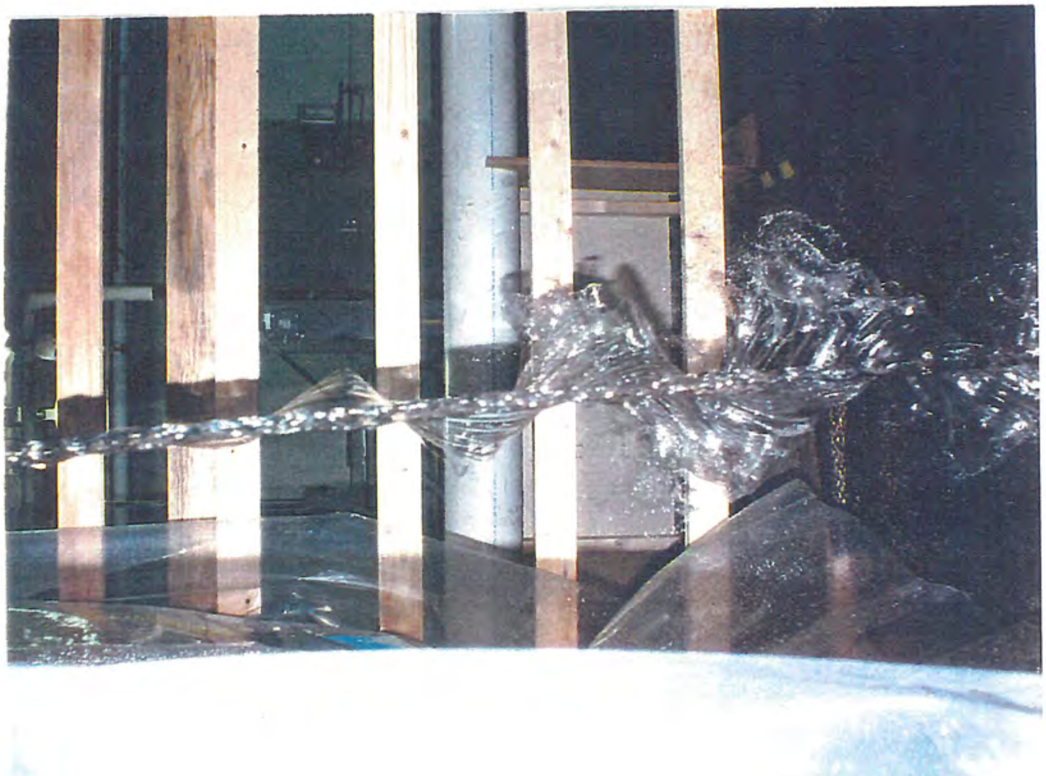

Figure 12. - Side view photograph of free jet showing the onset of instability

Figure 13. - Oblique view photograph of horizontally discharged free jet showing the onset of instability and break up.

CONCLUSIONS

Based on the results of these investigations, it appears that Eq'n (19) closely predicts jet velocity as compared with experimental observations. However, the limitation of these results is realized in consideration of contraction effects which cannot be precluded in describing jet thickness. Never-the-less, the solution to the steady ideal fluid case, provides a good description of two-dimensional laminar free jet motion downstream of the contraction region where viscous effects are relatively unimportant. Thus, to describe jet thickness in the contraction region, additional techniques are required and will be explored. This will allow for a complete description of viscous free jet case in the absence of aerodynamic effects prior to the onset of instability. Having achieved this, aerodynamic effects and surface tension can then be considered and a stability analysis conducted.

The qualitative nature of instability can be described using photographic techniques. The results obtained here provide a unique perspective on instability and allow for formulation of an approach to describing this problem both physically and mathematically. Furthermore, photographic results provide some insight into the possible modes of instability.

Finally, a complete statistical analysis of experimental results will be required to ascertain the statistical significance of trends exhibited by the experimental results and improve their interpretation.

FUTURE WORK

This project will be extended in the future to investigations focusing on the stability aspects of the steady, two-dimensional, laminar, viscous, free jet discharged horizontally into air. The first step will be to improve the analytical and experimental approaches identified in this report. This will include the application of mathematical techniques in describing contraction phenomenon near the crest for an improved analytical description of motion. In addition, the physical model will be improved to investigate a larger range of flow depth, H, experimentally. The results of which will be compared to the improved analytical description of motion. Ultimately, a stability analysis will be conducted in an attempt to describe and understand the onset of instability and breakup. Again, experimental observations will be obtained using the existing physical model for comparison with aspects of the stability analysis.

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