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FLOW FORMULAS FOR TURBULENT FLOW

Abflußformeln für Turbulente Strömungen

by

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Flow Formulas for Turbulent Flow

Willi H. Hager

Abstract

Flow in conduits, as well as normal flow in open channels, is today usually computed using the Darcy-Weisbach formula, together with the Colebrook-White equation for the coefficient of hydraulic resistance. The structure of this system of equations has been greatly modified compared to classical flow formulas. The justified question arises as to what extent these approaches coincide with each other. In addition to addressing this fundamental concern, the following study also proposes additional formulas to approximate virtually smooth and virtually rough flow regimes. The discussion specifically defines the scope of application for these formulas.

1. Introduction

In addition to determining flow rate, the calculation of pressure loss in conduits and normal flow in open channels represents one of the oldest problems posed by modern hydraulics. Quantitative analysis originated in the 18th century with Couplet's work on pipes and Brahms' study of open channels [3]. Since their day, numerous formulas have been introduced representing a relationship between the flow rate and the slope of the total head line (respectively, of the energy line or bed gradient), the pipe diameter (resp. the hydraulic radius), and the nature of the surface along the current boundary. Intensive efforts to measure and physically investigate flow rates culminated in the universally recognized Colebrook-White equation. When coupled with the Darcy-Weisbach formula, the resulting equation system is recognized today as the most accurate basis for calculating flow. It enables us to calculate flow rate in prismatic pipes or channels under stationary flow behavior, provided that the medium in question can be viewed as a Newtonian fluid, and the roughness characteristic remains locally unchanged. Given sufficient distance in open channels, normal flow prevails [6].

The phenomenon designated as the uniform flow condition represents the basic condition for fluid movement. So far as practical applications are concerned, this area of hydraulics is today considered to be a closed book. Nevertheless, dissemination of the Colebrook–White formula has actually given rise to a certain uncertainty on the part of engineers in the field. The classical flow formulas are much simpler than the Colebrook–White equation. In contrast to this simplicity, a "foreign element" is introduced into what was previously a fairly trivial calculation procedure. Although it was introduced almost fifty years ago, the Colebrook–White formula has not been totally accepted even today in the area of open channel hydraulics. Consequently, losses due to friction in uniform currents are seldom included in the calculation, or the formula is even rejected out of hand [11]. To what can we attribute this continuing aversion to the acceptance of a formulaic approach that is derived from exact physical principles and that has been verified most precisely? If the results obtained using this framework are more exact, what are the differences between them and those obtained using the classical approach? Which of the classical formulas, if any, may still be used? Unfortunately, little attention has been paid to these questions until now.

Kirschmer [9] has made a significant contribution toward the broad introduction of the Colebrook–White formula in the German–language community. To begin with, he notes that at the beginning of this century, more than one hundred flow formulas existed, producing chaos in this area. He cites in particular the relations proposed by de Chezy, Darcy–Weisbach, Gaukler–Manning–Strickler, Kutter and Bazin. Based on a critique of the last two formulas, he demonstrates that they do not satisfy the physical conditions required of them.

As early as 1961, Garbrecht [5] observed that in the rough flow range, the Manning–Strickler formula represents a good approximation of the Darcy–Weisbach formula when combined with the Colebrook–White friction coefficient, provided that the relative roughness ($\varepsilon = k_s/4\,r_{hy}$) lies in the range $5 \cdot 10^{-4} < \varepsilon < 5 \cdot 10^{-2}$. In a remarkable study, Dallwig [2] compares the formulas of Bazin, Kutter and Manning–Strickler with the law of universal flow. He concludes that all three comparative formulas are only applicable to the rough zone, hence apply only to open channels. However, Dallwig maintains that the roughness coefficients m according to Kutter and γ according to Bazin do not include any variable for the roughness of the channel wall. Since it can be demonstrated that the Ganguillet–Kutter formula for the slope $J_s > 1\%$ is subsumed in Kutter's formula, these three formulas will not be further discussed. The Manning–Strickler formula, on the other hand, yields results in the relative roughness range of $5 \cdot 10^{-4} < \varepsilon < 10^{-1}$ with an approximate degree of accuracy of 10%.

Finally, Unser and Holzke [12] compare the Colebrook–White formula with numerous approximation formulas that have been introduced either for the rough, turbulent range and the smooth range, or for the transition range between the two. Their work reveals that the cited relation generally provides the most accurate results, which is also why it forms the basis for the current study as well.

The goal of the following study is to compare most of the empirical flow formulas that have been introduced using the Darcy–Weisbach and the Colebrook–White system of equations. As a result, exponential formulas are proposed for virtually smooth and virtually rough flow systems, such that their variation from the reference system is negligible. Great attention is paid to the definitive ranges governed by the respective relations, and examples are provided of typical applications.

2. Inventory of Flow Formulas

2.1 Hydraulically Smooth Flow System

Flow rate formulas are defined as equations that demonstrate the relationship between the slope of the energy line J_E and the average velocity of flow v in closed conduits. In open channels, J_E is replaced by the slope of the stream bed, in which case, the corresponding relations describe the normal flow condition. Generally speaking, the transition relation J_E --> J_s applies with reference to the critical slope, and D --> 4 r_{hy} applies with D as the pipe diameter and r_{hy} as the hydraulic radius. If we ignore the influence of the form factor, flow in pressurized conduits and open channels can be described using the same equations. Hence, the following discussions refer to pressurized conduits, but can also be applied accordingly to open streams.

According to Forchheimer [3], Couplet was the first to deal with the determination of pressure loss in pressurized conduits in 1732. He was followed by Bossut, who set the average velocity proportional to the root of the pressure gradient. These French researchers were followed by Woltmann, Eytelwein, Weisbach and Gaukler. All their experimentally based relations combine v, J_E and D; the simplest type of formula can be represented as the ***exponential product***

$$v = C \cdot J_E^{\alpha} \cdot D^{\beta} \tag{1}$$

with $C[m^{1-\beta} s^{-1}]$ as the proportionality constant. ***Table 1*** summarizes various propositions. The table reveals that $1/2 \leq \alpha \leq 0.59$ and $1/2 \leq \beta \leq 0.765$; generally Reynolds found that $\alpha = (1 + \beta)/3$.

Author		C	α	β
Woltmann	(1791)	45,8	4/7	4/7
Eytelwein	(1796)	25,1	1/2	1/2
	(– –)	30,5	18/35	18/35
de Saint-Venant	(1851)	51	7/12	7/12
Dupuit	(1865)	25,5	1/2	1/2
Lampe	(1873)	54,2	5/9	25/36
		1)	1/2 to 0,59	1/2 to 0,765
Tutton	(1889)	25 to 35	0,51	0,66
Flamant	(1892)	68 to 75	4/7	5/7
Hazen & Williams	(1902)	56	0,54	0,63
Saph & Schoder	(1903)	74	4/7	5/7
Unwin	(1907)	2) 37,6	0,51	0,599
		3) 23,2	0,50	0,58
Foss	(1908)	50,35	6/11	8/11
Blasius	(1912)	$10{,}57\ \nu^{-1/7}$	4/7	5/7
Beyerhaus	(1920)	(–)	0,60	0,70
Wegmann & Aeryns	(1925)	49,1	7/13	0,723
Scobey	(1930)	37,1	9/17	108/187
Ludin	(1932)	~ 52	0,54	0,68
Stucky	(1943)	57,25	5/9	0,645
Scimemi	(1951)	61,55	0,56	0,68

Table 1: *Tabular listing of several exponential formulas according to [1], [3], including author, the year of publication, $C[m^{1-\beta} s^{-1}]$, as well as the exponents α and β. 1) For α and β resp. 2) For new cast pipe 3) For old cast pipe*

Because all measurements were conducted with water, C in Table 1 corresponds to a numerical value. Only Blasius' formula includes the kinematic viscosity ν; it has been checked both for water at various temperatures and for other media.

Only Unwin has considered different piping materials. Consequently, the proposed formulas relate to the specific type of pipe used in the tests and can only be applied to other types of pipe with care.

In addition to the type (1) formulas, complicated equations of the form

$$D J_E = a \cdot v + b \cdot v^2 \tag{2}$$

have been proposed, where a and b are functions of v and D. The approaches of de Prony (1804), Darcy (1858), Frank (1886), Lang (1889) and Biel (1907) have become familiar. However, these formulas are so complicated in comparison to Type (1) that they will not be discussed any further here.

Author		α	β	
de Chezy	(1818)	1/2	1/2	
Lahmeyer	(1845)	2/3	2/3	
de Saint-Venant	(1851)	11/21	11/21	
Humphreys and Abbot	(1861)	1/4	1/2	
Gaukler	(1868)	1	4/3	1)
		1/2	2/3	2)
Hagen	(1876)	1/5	1/2	3)
		1/5	1	4)
		1/2	2/3	5)
Manning	(1890)	1/2	2/3	
Forchheimer	(1903)	1/2	7/10	
Christen	(1903)	1/2	5/8	
Hermanek	(1905)	1/2	1	6)
		1/2	3/4	7)
		1/2	3/5	8)

Table 2: *Tabular listing of several exponential formulas for open channels according to [3], including the author with the publication date, as well as the exponents α and β according to Eq. (1);*
1) J_s < 0.07%, 2) J_s > 0.07%, 3) *for rivers,* 4) *for small streams,* 5) *for large, regular channels,* 6) h = < 1.5 m, 7) $1.5 \leq h \leq 6$ m 8) h > 6m

2.2 Hydraulically Rough Flow Regimes

De Chezy appears to have been the first to propose Equation (1) using $\alpha = \beta = 1/2$ for the normal flow condition in open channels. According to Eytelwein, the proportionality coefficient is $C = 25.4\ m^{1/2} s^{-1}$.

Table 2 shows additional propositions. With the exception of Humphreys and Abbot's formula and one of Hagen's proposals, α is always greater than or equal to 1/2, while β varies between $1/2 \leq \beta \leq 4/3$. It is apparent that all the newer formulas use $\alpha = 1/2$, while they propose that $\beta \approx 0.7$.

3. Modern Initial Approximation

3.1 Basic Relations

Today flow formulas are usually used to solve special problems, such as backwater and drop-down curves. In order to calculate pressure losses in conduits, however, the Darcy-Weisbach formula has established itself

$$J_E = \frac{v^2}{2g} \cdot \frac{\lambda}{D} \tag{3}$$

where λ is the coefficient of hydraulic resistance. As a result of turbulence theory, λ is a function of relative roughness $\varepsilon = k_s/D$ and the Reynolds number ($Re = (v \cdot D)/\nu$). For turbulent flow ($Re > 2300$) in pipes with commercial grade roughness characteristics, λ can be expressed using the Colebrook-White formula:

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log \left[\frac{\epsilon}{3{,}7} + \frac{2{,}51}{Re\sqrt{\lambda}}\right] \tag{4}$$

If $(\varepsilon/3.7) \ll [2.51/(Re\sqrt{\lambda})]$, we speak of a hydraulically smooth flow system, but if $(\varepsilon/3.7) \gg [2.51/(Re\sqrt{\lambda})]$, flow is in the hydraulically rough zone. If the influence of both terms is of about the same magnitude, we speak of the transition zone. It is important to note that both special cases, specifically

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log \left[\frac{2{,}51}{Re\sqrt{\lambda}}\right], \quad \epsilon \to 0; \tag{5}$$

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log \left[\frac{\epsilon}{3{,}7}\right], \quad Re \to \infty \tag{6}$$

describe asymptotic expressions of Eq. (4). They do not actually occur in nature, but are closely approximated, which is why the terms "virtually smooth"* and "virtually rough"* flow are used. According to [7], these conditions prevail as soon as λ (or comparable parameters) deviate less than 1.5% from asymptotic conditions. If we assume even 10% deviation from λ according to Equation (4), the law for hydraulic resistance can only be described by equations (5) and (6). The transition zone is then contained in the virtually smooth and rough flow zones in simplified form. In such a case, comparison with empirical flow formulas is quite simple.

3.2 Virtually Smooth Flow Regime

λ is an implicit function of Re in Equation (5). An approximate designation in the form of an exponential formula would be:

$$\lambda = \frac{\bar{C}}{Re^{1/N}} \tag{7}$$

*German usage prefers the terms hydraulically smooth and hydraulically rough. The terms used here, virtually smooth and virtually rough, would indicate that a flow condition can also exist on the boundary to the transitional zone.

where $\bar{C}$ [–] is the proportionality constant and N > 1 is the exponent. Blasius' formula is based on C = 0.316 and N = 4, which approximates Eq. (5) for $2.3 \cdot 10^3 < Re < 2.5 \cdot 10^5$ by better than ± 5%. ***Table 3*** shows a compilation of additional $\bar{C}$ and N values and indicates the corresponding application area for Re [8].

Table 3: *$\bar{C}$ and N in Formula (7) with corresponding application ranges [8]. Variation from Eq. (5) less than ± 5%.*

N	4	5	6
$\bar{C}$	0.316	0.194	0.125
Application Range	$2.3 \cdot 10^3 < Re < 2.5 \cdot 10^5$	$5 \cdot 10^5 < Re < 10^7$	$10^7 < Re < 2 \cdot 10^8$

Combining equations (3) and (7) yields:

$$v = \left[\frac{2gJ_E D^{(N+1)/N}}{\bar{C}\nu^{1/N}} \right]^{N/(2N-1)} \tag{8}$$

where v is a function of J_E, D and ν. Coefficient comparison with Eq. (1) yields:

$$\alpha = N/(2n-1), \quad \beta = (1+N)/(2N-1) \text{ and } \quad C = [2g/(\bar{C}\nu^{1/N})]^{N/(2N-1)} \tag{9}$$

As shown in [8], α only varies by ± 3% on either side of the average value $\alpha_m = 5/9$, while β varies by ± 7% on either side of the average value $\beta_m = 2/3$. However, for a constant temperature T of the medium, C varies by ± 15% on either side of the average value. ***Figure 1*** shows both parameters N and C (water at T = 15°C) as a function of the Reynolds number. The function N=N(log(Re)) can be approximated adequately by the straight line

$$N = 1.055 \cdot \log(Re) - 0.47 = 1.055 \cdot \log(Re/2.78) \tag{10}$$

For $\bar{C} = \bar{C}$ (Re), the following applies in the range $10^4 < Re < 10^8$:

$$\bar{C} = (50/Re)^{1/5}. \tag{11}$$

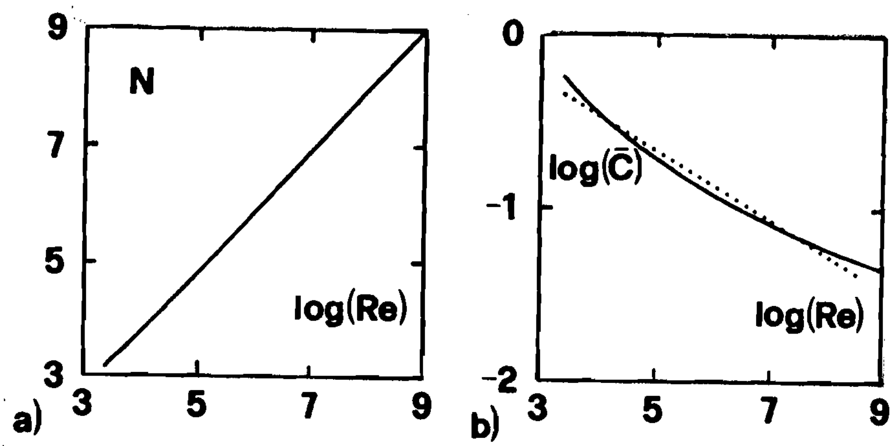

Figure 1: *Correlation between Eq. (5,7) smooth flow regime; a) N as a function of the Re number, b) $\bar{C}$ as a function of the Re number (...) Eq. (11)*

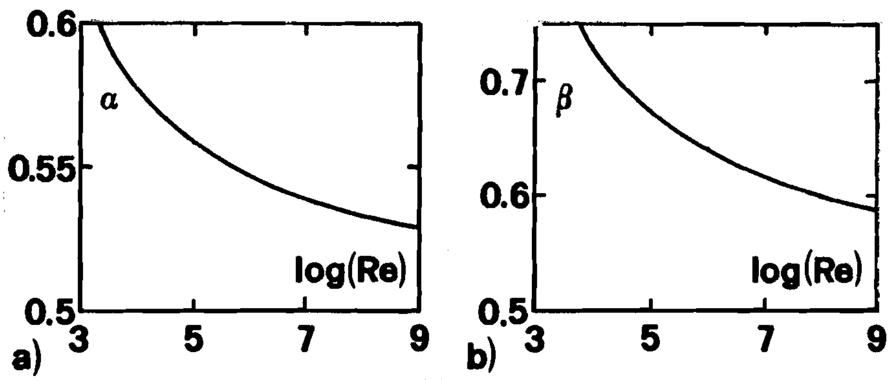

Figure 2: *a) α and b) β as functions of the Re number according to Eq. (9,10).*

Thus α, β and C can be expressed according to Equation (9) as functions of the Re number. As shown on ***Figure 2***, the ranges for α and β for $2.3 \cdot 10^3 < Re < 10^9$ are quite small. The following values apply: $0.53 < \alpha < 0.60$ and $0.59 < \beta < 0.75$. Both α and β vary in inverse proportion to the Re number.

Elimination of the Re number from both representations on Figure 2 yields a direct relation between α and β, as represented on ***Figure 3a***. The relation conforms to the Reynolds' formula

$$\alpha = (1 + \beta)/3 \qquad (12)$$

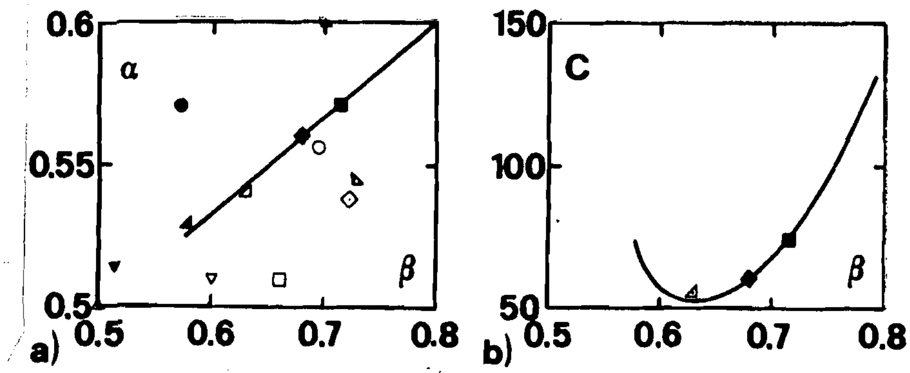

Figure 3: *a) (——) relationship between the exponents α and ß according to Eq. (12) and to (■) Woltmann, (△) Eytelwein, (▼) de Saint-Venant, (○) Flamant, Lampe, and Blasius, (□) Tutton, (●) Saph & Schoder, (▽) Unwin, (◣) Foss, (◬) Hazen & Williams, (◣) Beyerhaus, (◇) Wegmann & Aeryns, (▲) Scobey, (◆) Scimemi b) C as a function of ß according to Eq. (9-11) and to Table 1 (selected values)*

The combinations of α and ß suggested in Table 1 are also plotted in Figure 3a, indicating that in particular, Saph & Schoder's formula for $Re \approx 10^4$, Scimemi's for $Re \approx 10^5$, Hazen & Williams' for $Re \approx 10^6$, and Scobey's for $Re \approx 10^8$ represent excellent approximations of Eq. (5).

If we assume that the formulas are based on temperature T = 15°C, thus that $\nu = 1.145 \cdot 10^{-6} m^2 s^{-1}$, we arrive at the relation C(ß) shown on ***Figure 3b*** according to Eq. (9) and (11). The minimum for this equation is $C_{min} = C(ß \approx 0.63) \approx 52$. The four empirical approaches cited earlier are plotted on this curve based on Table 1. With the exception of Scobey's formulas, these values again conform well to results derived using the Colebrook-White formula.

Based on practical application, it would be desirable to cover the smooth flow regime with a single exponential formula. To do this, it is, of course, necessary to accept certain variations from Eq. (5). This kind of formula must be simply structured to use it with advantage together with the Colebrook-White relation and is particularly applicable for rough estimation. Because the standard range for Reynolds numbers is $10^4 < Re < 10^7$, the exponent N=5 should be used per Table 3. When $\bar{C} = 0.194$, we get a tangential approximation of Eq. (5) in this Re range. For smaller and larger Re numbers, however, λ is underestimated. In order to account for this circumstance, $\bar{C}$ is increased slightly, yielding the following proposition:

$$v = (10 g J_E)^{5/9} D^{2/3} \nu^{-1/9} \qquad (13)_1$$

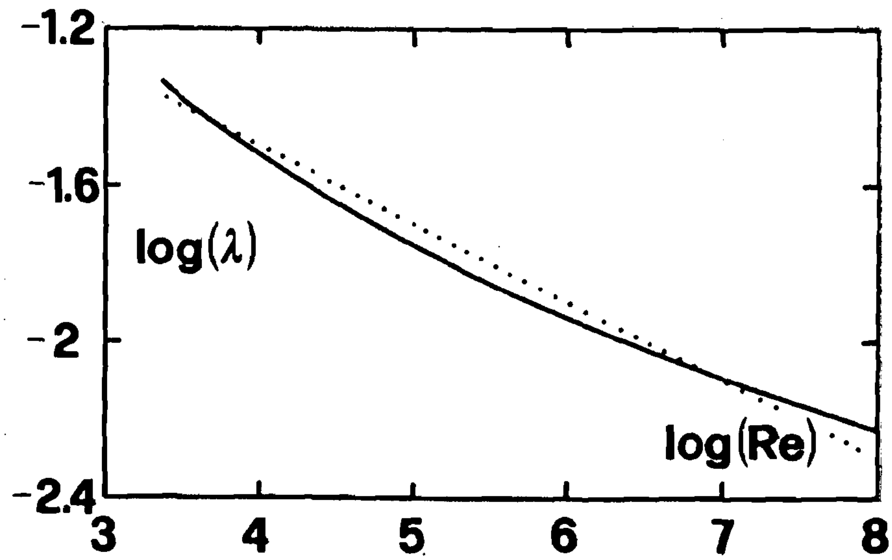

Figure 4: *Hydraulically smooth flow regime, λ as a function of the Re number according to Eq. (5) (——) and (···) Eq.* $(13)_2$

It is important to note that where $\alpha = 5/9$ and $\beta = 2/3$, Eq. (12) is satisfied. ***Figure 4*** shows a comparison between

$$\lambda = \frac{0.2}{Re^{0.2}} \qquad (13)_2$$

according to Eq. $(13)_1$ and Eq. (5). It is evident that λ is slightly over-estimated in the range of $10^4 < Re < 10^7$, i.e. when values (ν, Q, D) are given, J_E is also overestimated. In the range $2.3 \cdot 10^3 < Re < 2 \cdot 10^8$, the maximum variations between Eq. (5) and $(13)_1$ amount to ± 10%.

However, if (J_E, D, ν) are given and we solve for the flow rate

$$Q = \frac{\pi}{4}(10gJ_E)^{5/9} D^{8/3} \nu^{-1/9} \qquad (13)_3$$

we arrive at maximum variations of ± 5% in comparison to the Colebrook-White approach. Finally, if (Q, J_E,) are given and we solve for the diameter D, we approximate the formula

$$D = [4Q\nu^{1/9} / (\pi(10gJ_E)^{5/9})]^{3/8} \qquad (13)_4$$

with effective relations of better than ± 2%. This degree of accuracy, measured against the determination of other critical parameters, is probably too high. Finally, we should also note that flow is in the virtually smooth range as soon as [7]

$$k_s < \left[\frac{1.31\nu D_o}{Q}\right]^{8/9} * D_o, \text{ where } D_o = \left[\frac{Q^2}{gJ_E}\right]^{1/5} \tag{14}$$

Example 1: Given $Q = 10 m^3 s^{-1}$, $J_E = 0.01$, $k_s = 10^{-5}$ m, $\nu = 1.15 * 10^{-6} m^2 s^{-1}$; we solve for diameter D.

Where $D_o = [10^2/(9.81*0.01)]^{1/5} = 4.0$, it follows that $(1.31\,\nu * D_o/Q)^{8/9} * D_o = (1.31*1.15*10^{-6}/10)^{8/9}*4 = 1.18*10^{-5} m > k_s$, which places the flow in the virtually smooth range. Consequently, we can use Eq. $(13)_4$ here. Thus the following applies for the diameter: $D = [4 * 10 * (1.15 * 10^{-6})^{1/9}/(\pi(10 * 9.81 * 0.01)^{5/9})]^{3/8} = 1.474$ m. According to Eq. (4) this yields D = 1.494 m, which is 1.3% greater than the value found using the simple approximation equation.

3.3 Virtually Rough Flow Regime

If we solve Eq. (3) for the flow rate $v = \frac{\sqrt{2gDJ_E}}{\sqrt{\lambda}}$ and compare this expression with equations (1) and (6), it is evident that in the virtually rough range α must equal 1/2. (According to Eq. (6), λ is independent of J_E). β, on the other hand, is a function of the relative roughness ε. Because the flow rate decreases as the relative roughness increases, the following makes sense:

$$v = \frac{\sqrt{2gJ_E D}}{(\Phi\epsilon)^{1/2M}}, \text{ thus } \lambda = (\Phi\epsilon)^{1/M} \tag{(15)$_1$ and (15)$_2$}$$

It should be noted that although λ is explicitly a function of ε according to Eq. (6), the implicit system of equations (1, 6) should be solved to determine, for instance, the diameter D.

If instead of Eq. (6), we use

$$\lambda = [-2 \cdot \log(\epsilon/3{,}7)]^{-2} \tag{16}$$

and calculate the function $\lambda(\varepsilon)$, it becomes apparent that one cannot describe the rough flow regime using a single combination of Φ and M according to Eq. $(15)_2$ (***Figure 5***).

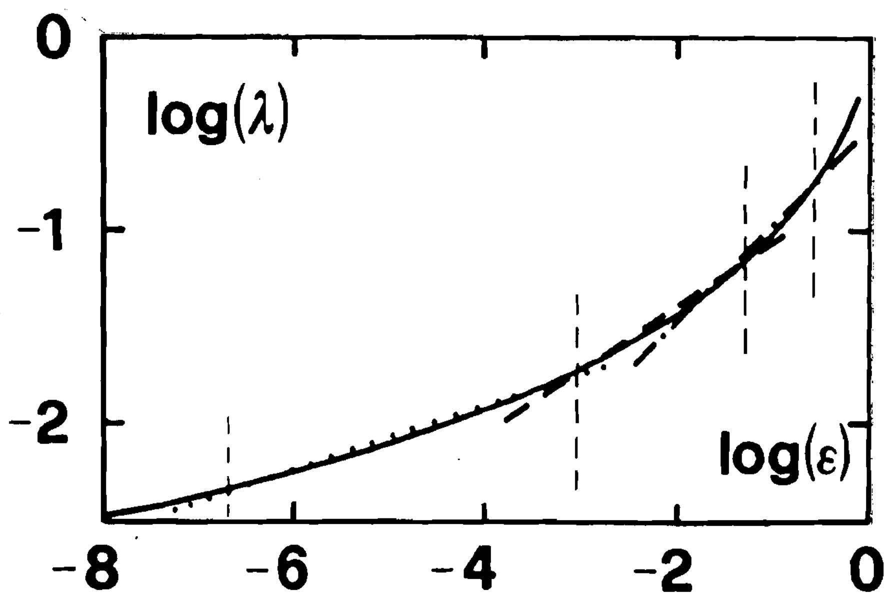

Figure 5: *Coefficient of hydraulic resistance λ as function of ε per (——) Eq. (6), (····) Eq.* $(18)_2$, (---) Eq. $(18)_1$ and (-··-) Eq. $(18)_3$. *Vertical broken lines show resp. ε-ranges.*

Through derivation of the two-sided logarithmic equation (16), it follows that

$$M = 1.151\,[0.568-\log(\varepsilon)]. \tag{17}$$

Table 4 shows a magnitude of M for various ranges of ε; Φ is also entered for Eq. $(15)_2$

log (ε)	-7	-5	-3	-1
M	~9	~6	~4	~2
Φ	$\sim 5*10^{-15}$	$\sim 2.5*10^{-8}$	$\sim 10^{-4}$	$\sim 10^{-1}$

Table 4: *M and Φ according to Eq. (6) and (15)2 for the rough flow regime.*

If we set the expression $\dfrac{\sqrt{2g}}{\Phi^{1/2M}}$ in Eq. $(15)_1$ equal to C according to Eq. (1), then it follows that $\beta = 1/2 + 1/(2M)$. If $2 < M < 9$ according to Table 4, the possible value range is $5/9 < \beta < 3/4$. Hence Gaukler, Hagen, Manning (all three with

$\alpha = 1/2$, $\beta = 2/3$), Forchheimer, Christen and Harmenek's equations as shown in Table 2 represent reasonable approaches.

The combination $\alpha = 1/2$, $\beta = 2/3$ is a special interest, in which case $M = 3$, and according to Eq. (17), $\varepsilon \approx 10^{-2}$. Thus $M = 3$ characterizes relatively rough flow in the rough zone. If we accept variations of ± 5% with respect to Eq (6), then

$$\lambda = 0.187\ \varepsilon^{1/3} \tag{18$_1$}$$

applies in the range $9 \cdot 10^{-4} < \varepsilon < 5 \cdot 10^{-2}$. According to the Moody Diagram, the appropriate range for the Re number is about $10^5 < Re < 10^8$.

For smaller values of ε, e.g. flow where $Re > 10^8$, compared to the Moody-diagram, M=7 applies. The relation

$$\lambda = 0.058\ \varepsilon^{1/6}. \tag{18$_2$}$$

varies less than 6% from Eq. (6) in the range $2 \cdot 10^{-7} < \varepsilon < 9 \cdot 10^{-4}$. Finally, for greater values of ε, specifically $5 \cdot 10^{-2} < \varepsilon < 2.5 \cdot 10^{-1}$, we get

$$\lambda = 0.34\ \varepsilon^{1/2}. \tag{18$_3$}$$

Correspondingly, we get

$$\begin{aligned}
v &= 5{,}87(gJ_E)^{1/2} D^{7/12} k_s^{-1/12} , && 2 \cdot 10^{-7} < \epsilon < 9 \cdot 10^{-4} ; \\
v &= 3{,}27(gJ_E)^{1/2} D^{2/3} k_s^{-1/6} , && 9 \cdot 10^{-4} < \epsilon < 5 \cdot 10^{-2} ; \\
v &= 2{,}42(gJ_E)^{1/2} D^{3/4} k_s^{-1/4} , && 5 \cdot 10^{-2} < \epsilon < 2{,}5 \cdot 10^{-1}
\end{aligned} \tag{19}$$

for flow velocity in pressurized pipes and

$$\begin{aligned}
v &= 13{,}18(gJ_s)^{1/2} r_{hy}^{7/12} k_s^{-1/12} , && 2 \cdot 10^{-7} < \epsilon < 9 \cdot 10^{-4} ; \\
v &= 8{,}24(gJ_s)^{1/2} r_{hy}^{2/3} k_s^{-1/6} , && 9 \cdot 10^{-4} < \epsilon < 5 \cdot 10^{-2} ; \\
v &= 6{,}85(gJ_s)^{1/2} r_{hy}^{3/4} k_s^{-1/4} , && 5 \cdot 10^{-2} < \epsilon < 2{,}5 \cdot 10^{-1}
\end{aligned} \tag{20}$$

for open channels.

If we compare these values with Table 2, Christen and Forchheimer's formulas approximate Eq. $(20)_1$ and $(20)_3$, while Manning's exactly reflects Eq. $(20)_2$. On the other hand, k_s represents the roughness characteristic in the new relations, a generally accepted value, and Eq. (19, 20) are dimensionally compatible. Furthermore, the roughness range is specified in which variations of less than 6% occur with respect to Eq. (6). Hence, J_E can be predicted with at least ± 6% accuracy when (k_s, D, Q) are known, resulting in variations of no more than ±3% with respect to the equation system (1,6). Finally, if (Q, k_s and J_E) are known and we solve for the diameter D or the hydraulic radius r_{hy}, then we can count on variations of less than ± 1.5% with respect to Eq. (1,6).

According to [7], we are in the virtually rough range if

$$k_S > [60 \cdot \nu D_0/Q]^{10/9} \cdot D_0 \tag{21}$$

where $D_0 = [Q^2/(gJ_E)]^{1/5}$. Application of the exponential formulas given above is thus related to two criteria: first it must be proven that flow is in the virtually rough range, and second, one of the three equations (19) or (20) must be applied, depending on the size of ε.

Example 2: In a rectangular channel with width b = 2 m, the following values are given: $Q = 10 m^3 s^{-1}$, $k_s = 5 \cdot 10^{-3}$ m, $J_s = 0.005$. What is the standard flow depth for water at temperature T = 15°C?

Where D_0 becomes $4 r_{hy0}$, $r_{hy0} = 0.25\,[100/9.81 \cdot 0.005)]^{1/5} = 1.148$ m. Since $k_s = 0.005 \text{ m} > [60 \cdot 1.15 \cdot 10^{-6} \cdot 1.148/10]^{10/9} \cdot 1.148 \text{ m} = 2.46 \cdot 10^{-6}$ m, flow is in the virtually rough zone. Based on Eq. $(20)_1$, we get the following for

$$Q = 13{,}18\ bh(gJ_S)^{1/2} \left(\frac{bh}{b+2h}\right)^{7/12} k_S^{-1/12} \tag{22}$$

thus yielding h = 1.488 m, compared with h = 1.570 m (+5%) according to Eq. (1,4), without accounting for the form coefficient; compare e.g. [2]. However, where $r_{hy} = hb/(b+2h) = 0.598$, thus $\varepsilon = k_s/(4r_{hy}) = 5 \cdot 10^{-3}$, flow is not in the rough range represented by Eq. $(20)_1$. According to Eq. $(20)_2$, it follows that

$$Q = 8{,}24 \cdot bh(gJ_S)^{1/2} \left(\frac{bh}{b+2h}\right)^{2/3} k_S^{-1/6} \tag{23}$$

thus yielding h = 1.573 m, compared with h = 1.570 m (−0.2%) according to Eq. (1,4), corresponding to

$$Q = -4\,bh\sqrt{2gJ_S r_{hy}} \cdot \log\left[\frac{k_s}{14.8\, r_{hy}} + \frac{2.51\,\nu}{8\sqrt{2gJ_S r_{hy}^3}}\right]. \qquad (24)$$

In conclusion, it should be noted that Eq. $(20)_2$ is also frequently applied in the form used by Manning−Strickler:

$$v = KJ_S^{1/2}\, r_{hy}^{2/3}\ . \qquad (25)$$

where K represents the friction coefficient and by virtue of coefficient comparison with Eq. $(20)_2$ is equal to

$$\frac{K \cdot k_s^{-1/6}}{\sqrt{g}} = 8.2\ . \qquad (26)$$

Hence if k_s is known, it is easy to determine $K[m^{1/3}s^{-1}]$.

4 Conclusions

The study described here has attempted to compare classical flow formulas with the results of modern turbulence theory and to derive appropriate propositions from this comparison. The Darcy−Weisbach formula serves as a basis for comparison, using the Colebrook−White formula to express the coefficient of hydraulic resistance. In special instances, the results reflect the virtually smooth and the virtually rough flow regimes. It is determined that a number of classic flow formulas represent excellent approximations of currently recognized relations, but it is noted that these exponential formulas are relatively recent (dating since about 1880).

If we accept variations of ± 10% with reference to the slope of the total energy line J_E (corresponding to ± 5% with reference to the flow or ± 2% with reference to the diameter), then it is possible to approximate the smooth flow regime using Eq $(13)_1$. It appears significant that the flow velocity v is a function of $J_E^{5/9}$ and of $D^{2/3}$. However, this formula can only be applied if inequality equation (14) is satisfied.

In the virtually rough flow regime, which is defined by inequality (21), three flow formulas are proposed, depending on the relative roughness ε (see Eq. (19) or (20)). For $9 \cdot 10^{-4} < \varepsilon < 5 \cdot 10^{-2}$, the proposition agrees with Manning–Strickler formula (25). The correlation between the equivalent sand roughness k_s and the friction coefficient K is expressed in Eq. (26). However, if ε is smaller or greater than the range indicated above, then the relations proposed in Eq. $(20)_1$ and $(20)_3$ should be used. However, the Manning–Strickler formula is particularly significant because the roughness range covered by this formula occurs most frequently in practical applications. It should be noted that in both the virtually smooth as well as the virtually rough flow regimes, v is a function of $D^{2/3}$, resp. $r_{hy}^{2/3}$.

Designated Variables

a [s]	Coefficient according to de Prony's formula
b [$s^2 n^{-1}$]	Coefficient according to de Prony's formula
b [m]	Surface width [e.g. Eq. 22, 23, 24]
C [$m^{1-\beta} s^{-1}$]	Proportionality factor in the flow formula
$\bar{C}$ [–]	Proportionality factor for the coefficient of hydraulic resistance
D [m]	Pipe diameter
D_o [m]	Reference diameter
g [ms^{-2}]	Normal case acceleration
h [m]	Water depth
J_E [–]	Slope of the total energy line
J_s [–]	Slope of the channel bed
k_s [m]	Equivalent sand roughness
K [$m^{1/3} s^{-1}$]	Manning–Strickler coefficient of roughness
N [–]	Exponent in the equation for hydraulic resistance
Q [$m^3 s^{-1}$]	Flow rate
r_{hy} [m]	Hydraulic radius
Re [–]	Reynolds number
v [ms^{-1}]	Flow velocity
α [–]	Exponent for the slope of the total energy line
β [–]	Exponent of the diameter
ε [–]	Relative roughness
λ [–]	Coefficient of hydraulic resistance
ν [$m^2 s^{-1}$]	Kinematic viscosity

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