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Ground-Water Flow by Relaxation Methods

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GROUND-WATER FLOW BY RELAXATION METHODS

P. "F" Enger*, M. ASCE

Introduction

A problem of recent interest to the Hydraulics Branch of the Bureau of Reclamation concerned the flow pattern occurring when a deep permeable aquifer containing salt water and overlain by a less permeable material was subjected to a surcharge, which resulted from irrigation with fresh water. The two aquifers were of approximately equal thickness, Figure 1, and the water table was relatively close to the soil surface.

It was desired to know the action of the fresh water-salt water interface when horizontal drains were placed at the ground-water elevation and a surcharge was applied to the ground water.

Several paths of investigation were followed, one of which was a scale hydraulic model study performed in the Hydraulics Branch.[1/] Mathematical studies were performed by R. E. Glover[2/] and have been reported. The model studies provided information on the movement of the salt water-fresh water interface with respect to time. The mathematical studies provided solutions for the final position of the interface for certain model conditions.

To determine the feasibility of tracking the interface with respect to time, by using an electronic digital computer and numerical procedures, relaxation methods were investigated. Relaxation methods may be described as

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[1/] Numbers refer to references listed at the end of the paper.

reiteration procedures similar to the moment distribution method used in analyzing statically indeterminate structures, or the Hardy Cross distribution method used for studying pipe systems. However, the relaxation method was developed independently, by Professor Southwell and his associates at Oxford University. These developments have been published in many papers but are presented principally in books by Southwell,[3/] [4/] [5/] Allen,[6/] and Shaw.[7/]

The principal advantage of the relaxation method lies in the fact that the solution of problems is obtained by simple and direct treatment; also the method may be applied to solve problems which are unsolvable by other methods. The method is approximate, but provides satisfactory results by reducing errors to assigned minimums.

To use relaxation principles, it is necessary to establish a mathematical model which can be operated on by numerical methods. For ground water flow, with which this paper is concerned, a finite difference equation which approximates the second order Laplace differential equation is used. The Laplace equation is generally accepted as a definition of the physical system of seepage conditions, but it usually cannot be integrated for complex problems where boundaries may not be simple geometric shapes.

To facilitate computation procedures by relaxation methods, the two-dimensional surface of interest is divided into small squares. Numerical values are then determined, by reiteration methods, at every line intersection. These numerical values represent the value of the function within some assigned margin of error. Although other types of divisions may be used only square divisions were used in this work. Within practical limits the smaller the square divisions are made the more accurate will be the results. Divisions may be of two sizes when increased accuracy is desired in certain areas.

Before electronic digital computers were generally available a considerable amount of hand labor was necessary to solve a relaxation problem. However, with computers being generally available, the method may find wider application as abilities to program computers increase and computers become larger and faster.

Problem Considerations

To obtain a solution for a differential equation regarding ground-water flow, the practice has usually been to make several simplifying assumptions. These assumptions usually include constant inflow, equilibrium conditions, constant and level terrain, and soil of one permeability. Ideal conditions are seldom found in nature. Actual conditions usually involve variable inflow such as water supplied by storms or irrigation over a given area and time, varying terrain, and several different soil strata.

A cross section of the condition of interest for this study is shown in Figure 1. The section consists of two layers of approximately equal thickness. The permeability of the lower layer of soil is greater than the permeability of the upper layer. Salt water stands in the lower layer and is overlain by fresh water to the elevation of the drainage system.

Figure 1 shows the system in a state of equilibrium. At some time, t_2, a volume of fresh water may be added as a surcharge to some portion of the water surface. This surcharge might, for instance, result from a storm or irrigation. At time, t_2, conditions may look similar to those shown in Figure 2a. The surcharge will result in a new set of pressure and flow conditions so that at time, $t_3 = t_2 + \Delta t$, conditions in the cross section may be as shown in Figure 2b. If no additional surcharge is added, the interface may be forced to some maximum displacement condition, at $t_i = t_2 + m\Delta t$, similar to that shown in Figure 2c. However, at some later time, t_j, as water flows from the drains and unbalancing pressures become less, new conditions similar to those shown in Figure 2d may be established, and finally, as $t_2 + m\Delta t$ becomes very great, an equilibrium condition somewhat similar to that shown in Figure 1 will again be established.

If during the preceding change another volume of water were to be added to this or another area, pressure and flow conditions would change in a different manner.

It appears then that for most natural conditions a salt water-fresh water interface of this type would usually be in a state of motion. The amount of motion would, of course, vary with the soil permeabilities, the water densities, the surcharge conditions, location of surcharges, time after application of surcharges, and several other variables.

Location of Fresh Water-Salt Water Interface--Stable Condition

If a constant rate of inflow were to continue for a very long time over a given area, the interface would reach a near stable condition. Carlson[1/]

and Glover[2] considered a condition of this type where the soil was of one permeability, Figure 3.

At the stable configuration the salt water below the interface would become near stagnant, and within the stagnant area the pressure at some distance y_1 below the drain must be

$$p_1 = y_1(\rho + \Delta\rho)\, g$$

where

ρ is the density of water above the interface

$\Delta\rho$ is the increase in water density below the interface which is due to the salt content

g is the acceleration of gravity

The same pressure occurring at the interface at this point may also be written as:

$$p_1 = \rho g\, (y_1 + h)$$

where h is the distance of the free water surface above the drain.

Equating p_1 results in:

$$y_1 = \frac{\rho}{\Delta\rho}\, h \qquad \ldots\ldots \quad (1)$$

Previous studies[8], [9], [10] have shown this relationship, referred to as the Ghyben-Herzberg relationship.

By assuming a continual and well distributed constant rate of inflow, Glover[2] develops the following equation to explain the position of the water surface:

$$h = \sqrt{\frac{i\,L^2}{4(1+m)k}}\;\sqrt{1-\left(1-\frac{2x}{L}\right)^2} \qquad \ldots\ldots \quad (2)$$

where

L is the drain spacing

x is the distance from the drain

i is the inflow

m is $\rho/\Delta\rho$

These equations were checked by model studies[1] and the position of the water surface and interface were found to be substantially as predicted by the equations, Figure 4.

To find how the change in density, permeability and inflow influence the position of the interface, a short computer program[11] was written. Results from the program are shown in Figure 5. Figure 5a indicates the influence of the density of the salt water. It will be noticed that as the density of the salt water approaches the density of the fresh water the displacement of the interface becomes greater. Figure 5b indicates the influence of the soil permeability. As the permeability becomes greater, the displacement of the interface becomes less. Figure 5c indicates the influence of inflow. As the inflow increases, the interface displacement increases. Figure 5d indicates the influence of the three factors on the maximum displacement of the interface, for the range of interest in this study. A study of the figure shows that for the ranges in question the slope of the inflow and density difference curves has no large changes. However, for small permeabilities, small changes in the permeability results in large changes in the displacement of the interface. Density differences which approach zero would also result in large changes in the interface position. However, these small differences were not of primary interest in this study.

The dimensions used for Figure 5 are similar to those used for the model study.[1/] The time required for the interface to reach the equilibrium positions is not considered.

Relaxation Procedures

Kirkham and Gaskell[12/] have used relaxation methods to determine the position of a falling water table in tile and ditch drains by considering a number of steady-state problems separated by small increments of time, Δt.

To determine the displacement of the salt water-fresh water interface with respect to time it appeared feasible to use a similar procedure to track the interface, as well as the falling water surface between drains. An electronic digital computer program which would provide the position of the interface and water surface at various times would permit many aquifer configurations to be studied. By using the program several configurations could be studied in a relatively short time, and the necessity of physically preparing and operating a model for every configuration of interest would be eliminated.

The problem was considered two dimensional and Laplace's equation[13] was used as the governing equation. Finite difference approximations were obtained from developments similar to those shown in references[3] through [7]. A simple development of residual and relaxation operators is given in the appendix.

A surcharge was placed on the stable configuration shown in Figure 1 and the unbalancing pressure condition was calculated by a program written (in FØRTRAN) for an electronic digital computer.[11] Seepage velocity was established in the program as

$$V = \frac{k}{p} \frac{\Delta h}{L}$$

where

- k is the permeability coefficient
- p is the porosity
- Δh is a change in head from an equilibrium condition for a given length, L.

Direction of the velocity may readily be established; for instance, a flow line is at right angles to an equipotential line.

After establishing the velocities, a relatively short period of time is used in the program for establishing the position to which the water surface and the salt water-fresh water interface would move under the existing pressure conditions. New residual and relaxation operators are calculated for the new position of the surface and interface, and the process is repeated. At any desired intervals the computer punches the position of the surface and interface into cards which may be used on an x-y plotter. Thus the position of the surface and interface may be plotted at any desired time.

Several assumptions were made. For instance, the problem is considered to be two-dimensional and capillary effects are neglected. Pressure transmission velocities are assumed to be much greater than water velocities, pressure heads are assumed large compared with velocity heads, and temperature is assumed constant.

Check of Stable Conditions

To determine if the computer program would determine an interface position comparable to a real condition, a check of the model reported by Carlson[1/] was conducted. Test 7, on which considerable data were presented, was used for the check. The computer program[11/] was revised slightly for this run, as it was desired to maintain a constant water surface. The water surface position was calculated and a subroutine to calculate a new position for the fresh water surface was not called. Necessary compensating statements were added at other points in the program.

A density difference of 0.070 pound second^2 per feet^4 used with a permeability of 0.032 foot^3 per foot^2 per minute and an inflow to the model of 1.566 by 10^{-4} feet^3 per feet^2 per minute results in the calculated interface, shown on Figure 4. This interface was located by use of the stable condition equations (1) and (2). The position of the model interface after 142 hours of operation and 310 hours is also shown.

Checks were made on the position of the interface at 53.5, 75.5, and 99.5 hours, Figure 6. These indicated that the model interface and the calculated interface were moving at about the same velocity, and that their positions were approximately equal at these times.

The computer program indicated some slight nonsymmetrical interface conditions especially near the drains. The nonsymmetrical conditions seem to occur because of the relatively large net used for the calculation (0.3 foot square), and from allowing the residual error (see appendix) to be fairly large. The residual error was allowed to be fairly large to reduce the necessary running time on an electronic digital computer. Approximately 30 minutes of running time on a large digital computer were required to track the interface for 99-1/2 hours of real time.

Results from the program were considered satisfactorily close to the model results, and additional investigations were conducted of a more complex model.

Investigations of Unsymmetrical Surcharge

Investigations were conducted to determine the influence of permeability changes of a two-layered aquifer on the flow pattern. Actual dimensions of a model available in the Hydraulics Branch were used, Figure 7. The model length was 16 feet; the horizontal drains were located 2.9 feet above the bottom of the aquifers; the thickness of the lower permeable layer was 1.4 feet; and the salt water stood at 0.9 foot above the bottom of the aquifer. The salt water was overlain with fresh water to the drain elevation.

On this stable configuration was superimposed an unsymmetrical surcharge of 0.4 foot of fresh water. The surcharge was superimposed at the same location for all tests. Permeabilities were varied as shown in Table 1.

Table 1

Test No.	Relative permeability of upper layer	Relative permeability of lower layer
1	k*	50k
2	k	10k
3	10k	10k
4	10k	50k

$k = 0.033\ ft^3/ft^2$ minute

The computer program[11] was used to track the interface and the water surface. The geometry, physical properties, and the surcharge were read into the computer. The change in water table and the interface position were determined for several time intervals. When desired, punched cards containing the value of the water surface and the interface were requested from the computer. The cards were used with an x-y plotter to plot the positions of the water table and the interface.

Results of the runs are shown in Figures 7, 8, 9, and 10. As will be noticed from a study of these figures, changes in the porosity of the aquifer layers have pronounced effects on the configuration of the fresh water-salt water interface. However, the general form of the displacement of the interface remains the same, regardless of the porosities of the aquifers, and the pronounced effects come from change in magnitude of the displacements.

On Figure 7, are shown results for relative permeabilities of k and 50k. The figure indicates that the fresh water surface in the area of low permeability moves slowly compared to the fresh water-salt water interface located in the layer of higher permeability. Directly below the surcharge the interface moves down, but generally moves toward the drains in other areas.

On Figure 8, are shown the results for relative permeabilities of k and 10k. The fresh water surface in the area of low permeability moves at nearly the same rate as that shown in Figure 7. The magnitude of movement of the interface has been considerably reduced, but is in the same general direction as that on Figure 7.

Figure 9, shows the results for relative permeabilities of 10k and 10k. Very little movement results at the interface. However, movement of the fresh water surface has increased considerably. Movement of the fresh water surface is similar to that shown in Figure 10, which shows the results for relative permeabilities of 10k and 50k. However, in Figure 10 considerable movement of the interface is indicated.

Total discharge from the drains was recorded as a portion of the computer output and is shown in Figure 11. Generally as the upper layer became more permeable the discharge from the drains increased. However, the lower layer did have some influence on the discharge. As indicated by the results of Tests 3 and 4 the discharge from the drains decreased as the permeability in the lower layer became greater.

Conclusions

This study indicates that considerable information on ground-water flows can be developed using electronic digital computers and relaxation procedures. For instance, the influence of (1) density differences, (2) additional loading, (3) three, four, or more soil layers, and (4) sloping soil layers, etc., can readily be determined.

If the physical conditions pertaining to the problem are complex the program may be long, and considerable machine time may be required to execute the program. A machine with a large core memory and fast execution time is desirable for this type of program. The programs may be difficult to check for accuracy of results, but to assure correct results are obtained, every attempt should be made to check them.

Southwell[5/] has shown that relaxation methods may be used to solve three-dimensional problems, thus indicating their tremendous possibilities in future work. It would appear that in future work the greatest limitations on the use of relaxation methods will be the users ability to establish the computer program, and the availability of digital computers fast and large enough to obtain the desired information.

The relaxation concept is a useful research tool, for the solution of ground-water problems, particularly when used with a digital computer.

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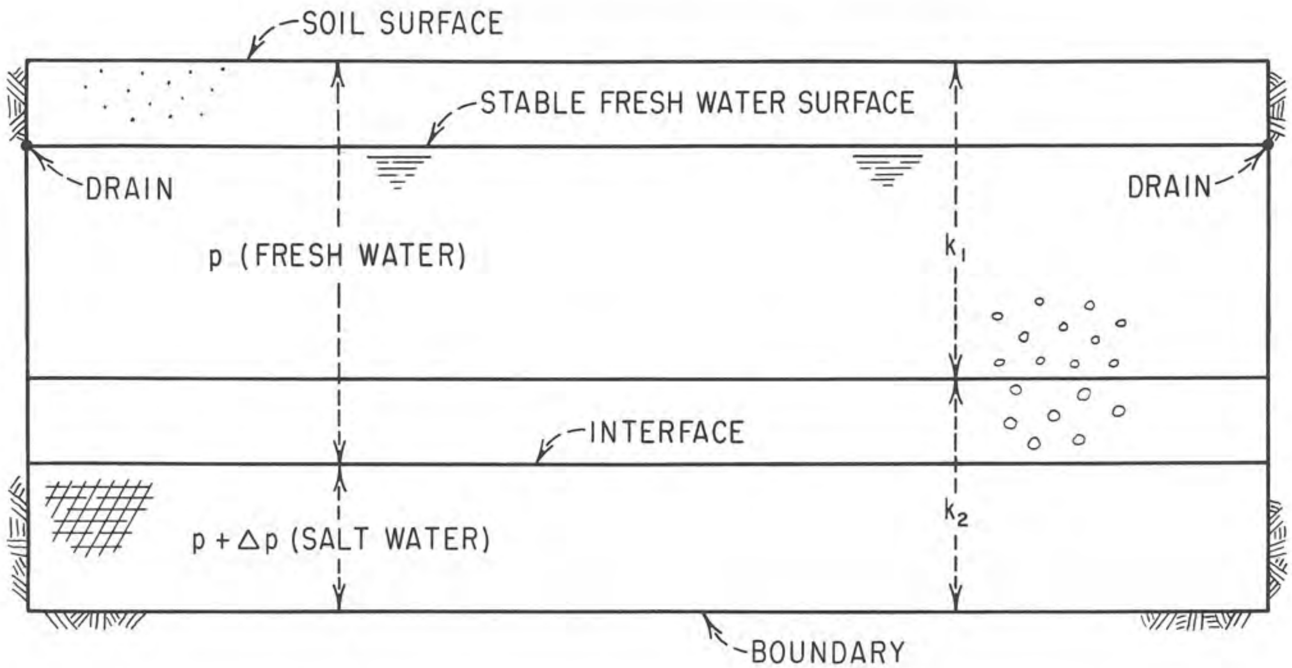
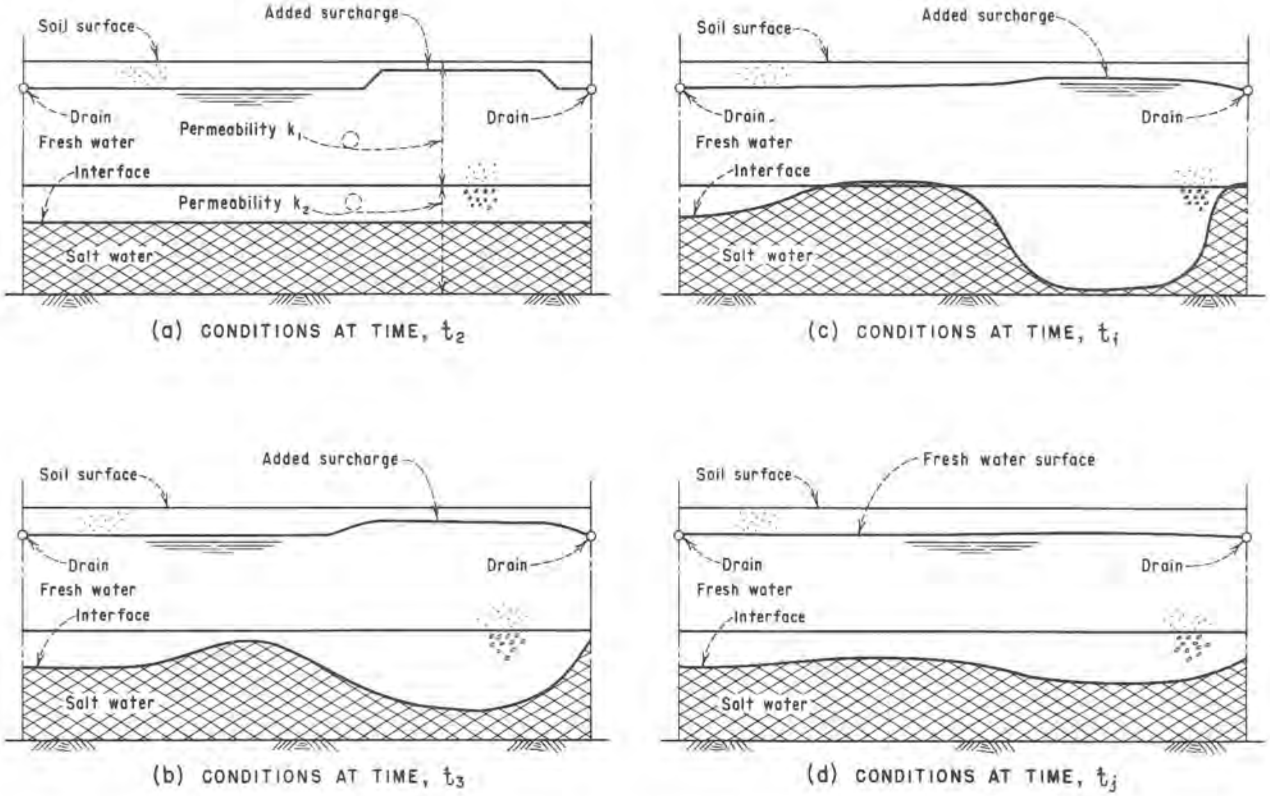


GENERAL CONDITION OF INTEREST

FIGURE 1



(a) CONDITIONS AT TIME, t_2

(c) CONDITIONS AT TIME, t_i

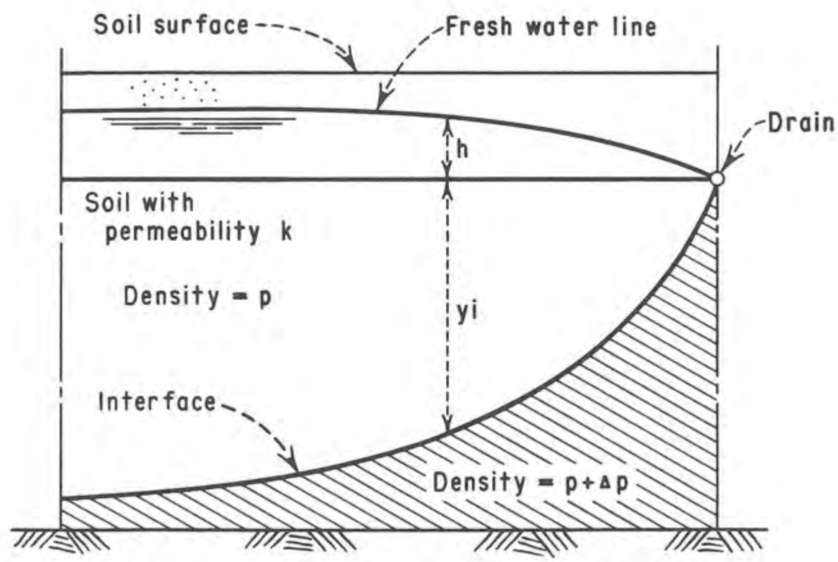
(b) CONDITIONS AT TIME, t_3

(d) CONDITIONS AT TIME, t_j

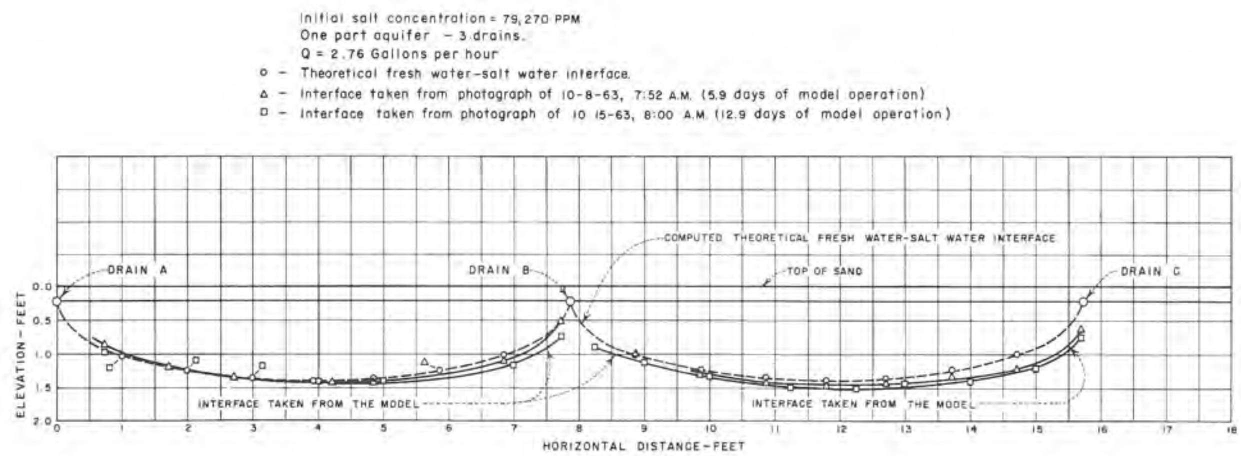
GENERAL INTERFACE MOVEMENT DUE TO SURCHARGE

FIGURE 2

FIGURE 3

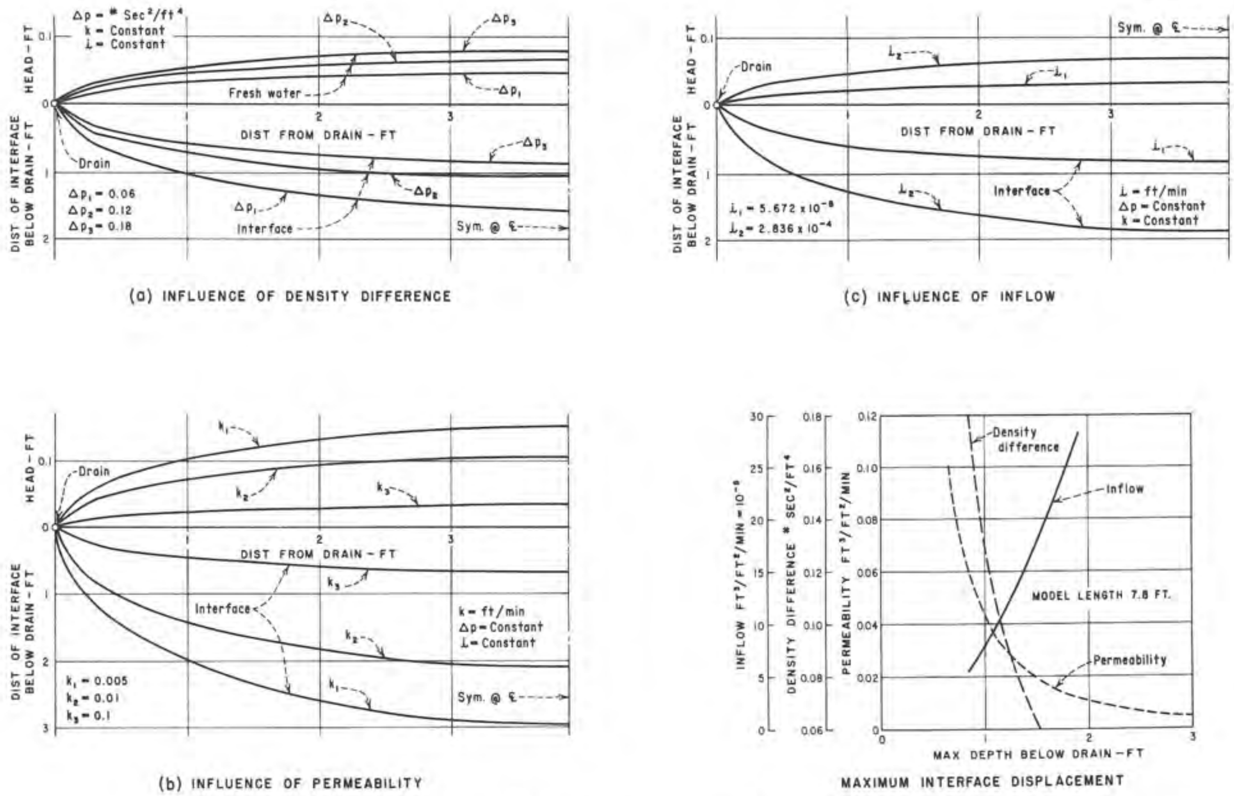


STABLE INTERFACE CONDITIONS



GILA VALLEY
GROUND WATER MODEL
TEST NO. 7

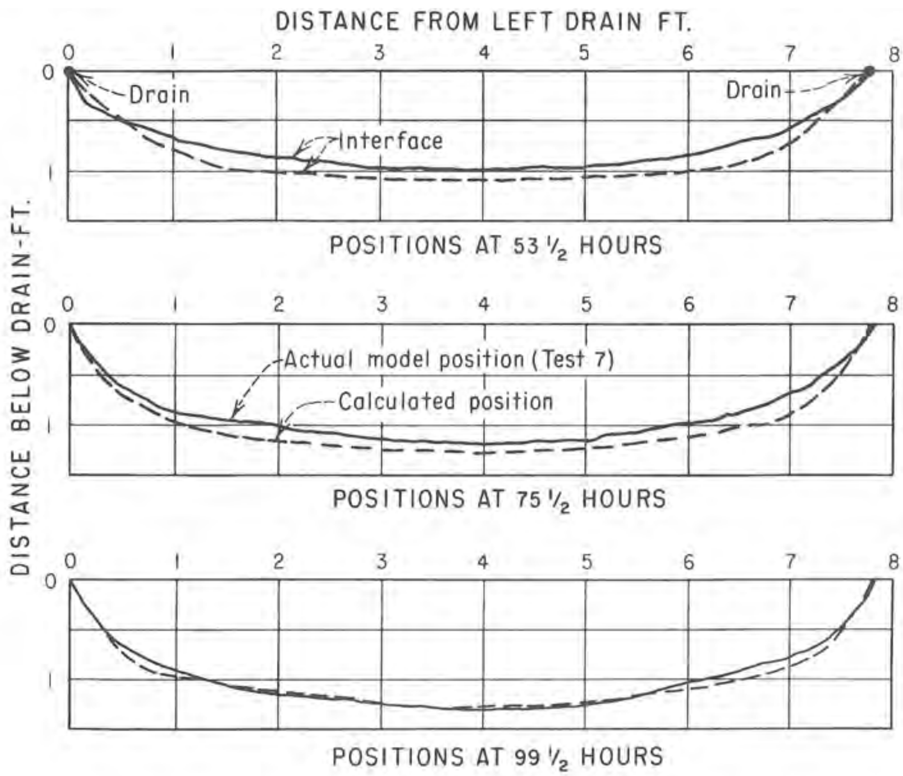
FIGURE 4



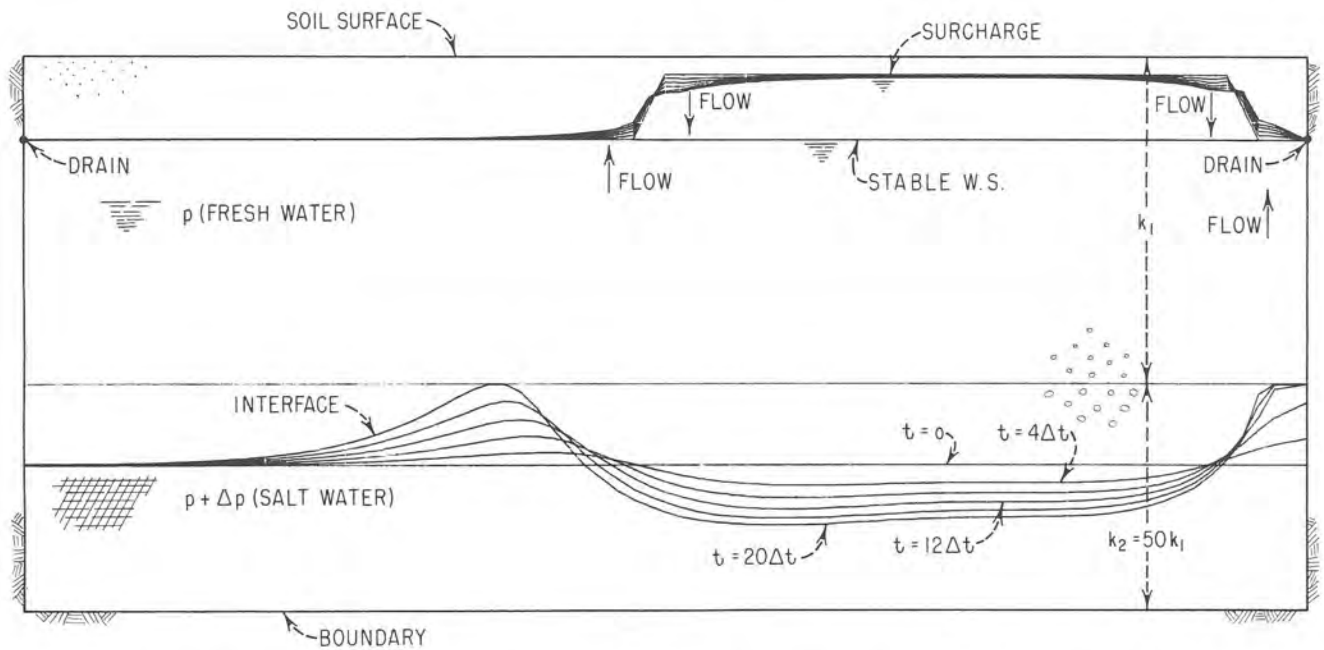
INFLUENCE OF DENSITY, PERMEABILITY AND INFLOW

FIGURE 5

FIGURE 6

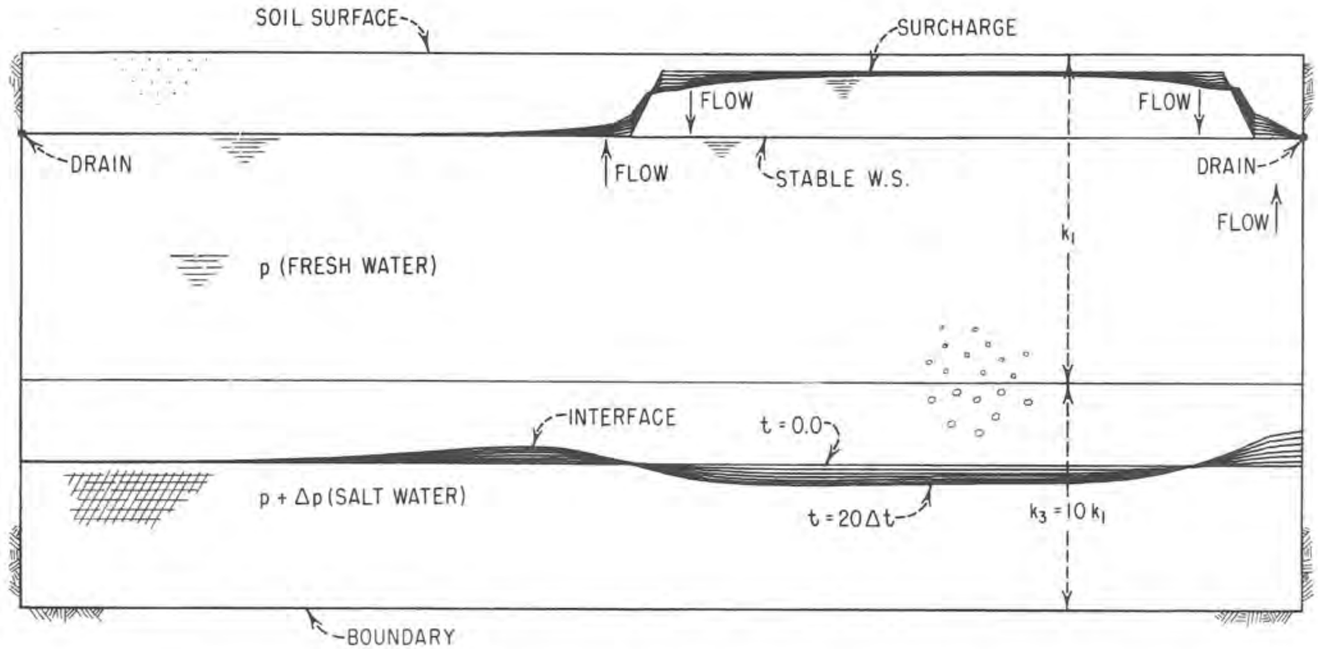


DISPLACEMENT OF A FRESH WATER – SALT WATER INTERFACE



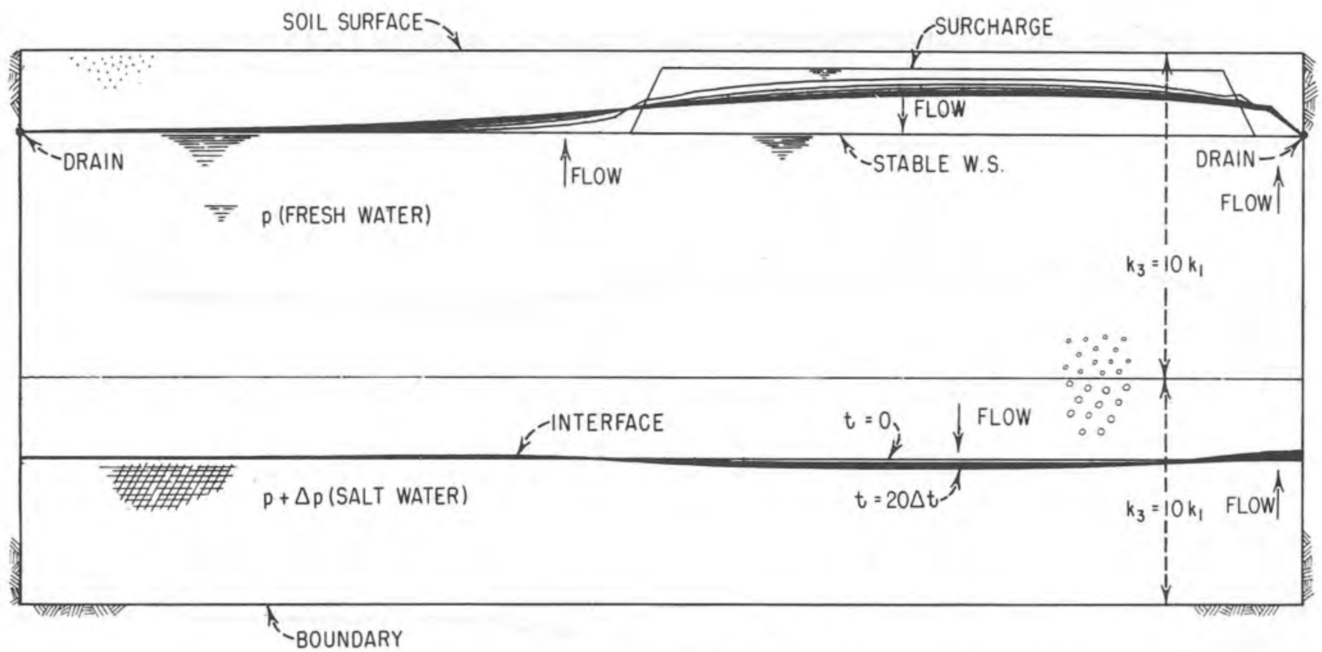
INTERFACE DISPLACEMENT FOR RELATIVE PERMEABILITIES OF k AND 50k

FIGURE 7



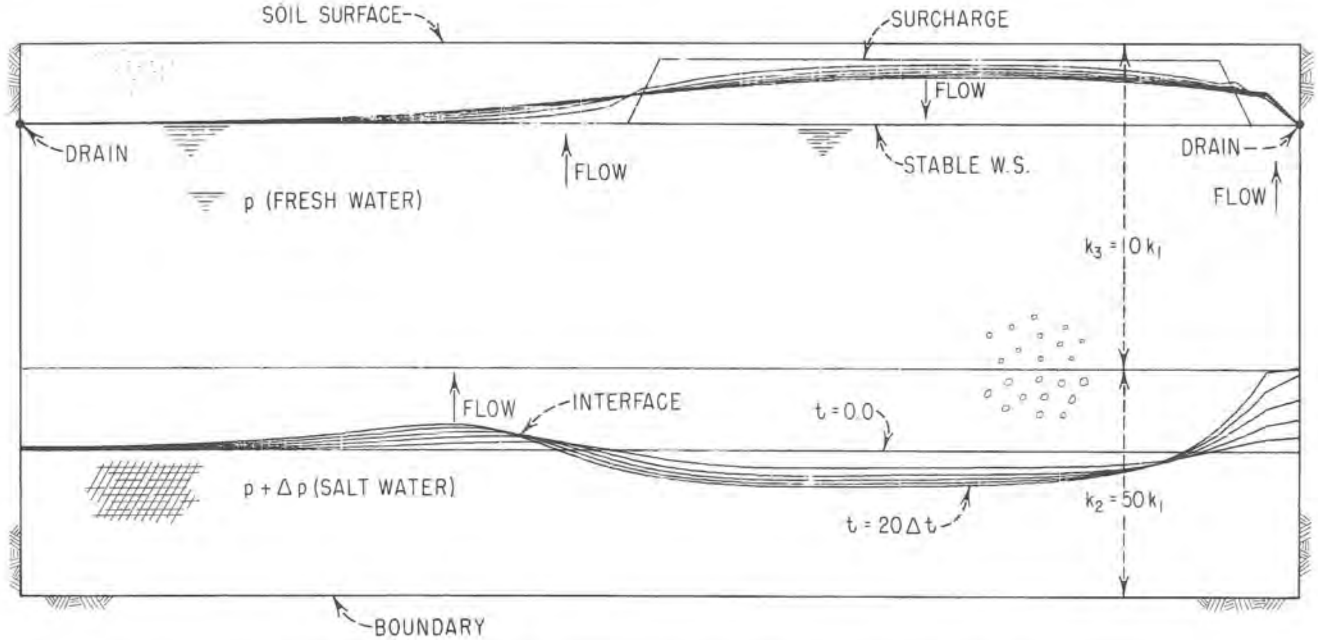
INTERFACE DISPLACEMENT FOR RELATIVE PERMEABILITIES OF k AND 10k

FIGURE 8



INTERFACE DISPLACEMENT FOR RELATIVE PERMEABILITIES OF 10k AND 10k

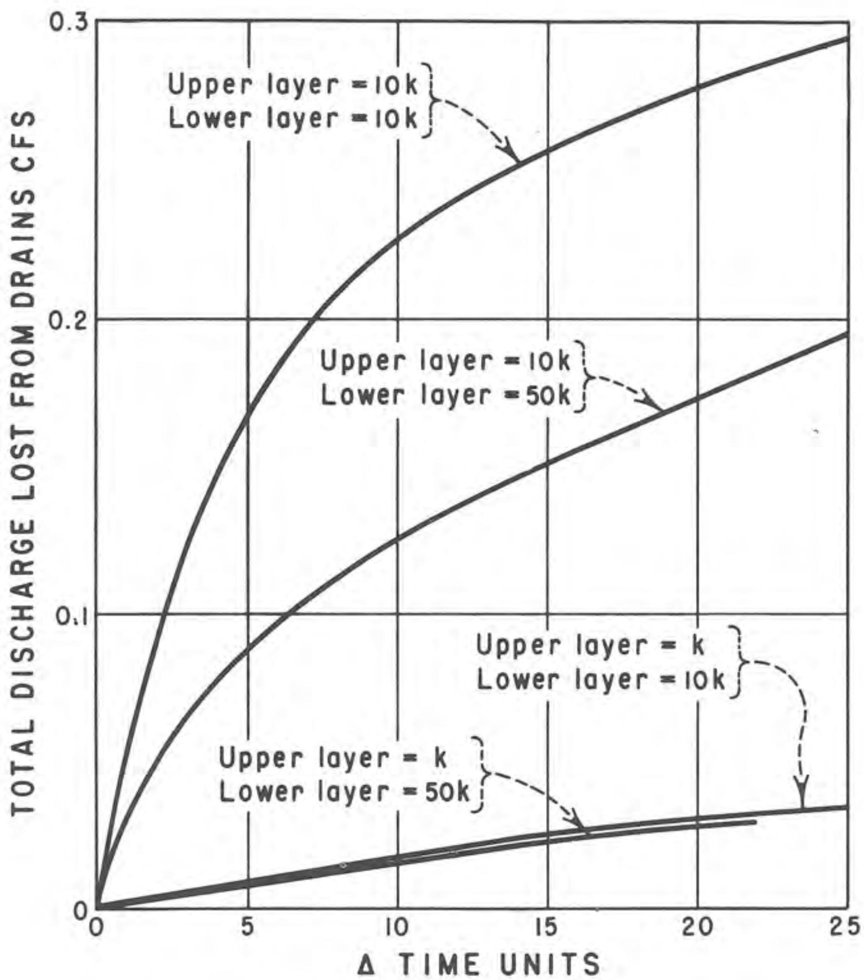
FIGURE 9



INTERFACE DISPLACEMENT FOR RELATIVE PERMEABILITIES OF 10k AND 50k

FIGURE 10

FIGURE II



DISCHARGE FROM DRAINS

APPENDIX

Some Information on Relaxation Procedures

Consider only the expressions of immediate interest

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2}$$

where $z = z(x,y)$.

Make the assumption that the function is continuous as the point (x,y) ranges over the domain of interest.

Consider a zone in the x,y plane divided into small squares by lines which are parallel to the x and y axis as shown in Figure 12a.

Each square has side length h, and to maintain a notation developed by Southwell,[1/] Allen,[2/] and Shaw[3/] the intersections of the lines have been numbered and may be referred to as nodal points.* Although the point 0 could be assumed to be any point in the plane, for convenience it is assumed to be the origin ($x = 0$, $y = 0$). For the value of $z = z(x,y)$ at the origin, assign the value of z_o. It should be noted that with Southwell's notation there is no connection between the number of the points and the values of x,y at the point. For a simple derivation of a finite difference expression (as h is a small quantity) consider the z,x plane at $y = 0$. The edge of the plane may have an appearance similar to that shown in Figure 12b.

[1/,2/,3/] Numbers refer to references listed at end of appendix.

*In developing the theory a net was envisioned as a replacement for a continuous film. The intersections were called nodal points of the net.

Consider the quantity $\frac{\Delta z}{\Delta x}$; as h becomes small, note that this quantity approaches $\frac{\partial z}{\partial x}_{y=o}$ or

$$\frac{\partial z}{\partial x} \underset{h \to o}{\to} \frac{f(x+h) - f(x)}{h} \qquad {}_{y=o} \qquad \ldots \quad (1)$$

and neglecting small errors, if h is small

$$\frac{\partial z}{\partial x} \approx \frac{z_1 - z_j}{h}$$

or at x_o, by utilizing 2h

$$\left(\frac{\partial z}{\partial x} \right)_o \approx \frac{z_1 - z_{-1}}{2h} \qquad \ldots \quad (2)$$

also as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$$

$$\frac{\partial^2 z}{\partial x^2} \approx \frac{\left(\frac{\partial z}{\partial x} \right)_{x+h/2} - \left(\frac{\partial z}{\partial x} \right)_{x-h/2}}{h}$$

where the half interval is used to result in the approximation for the second derivative being in terms of z_1, z_o, and z_{-1}.

By substitution:

$$\left(\frac{\partial^2 z}{\partial x^2} \right)_o \approx \frac{\left(\frac{z_1 - z_o}{h} \right) - \left(\frac{z_o - z_{-1}}{h} \right)}{h}$$

and

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{\substack{0\\y=0}} \approx \frac{z_1 - 2z_0 + z_{-1}}{h^2} \qquad \ldots (3)$$

Higher derivative could be found in a like manner. However, as these are the equations of immediate concern, no higher order derivatives need be established.

In a similar manner, a section in the z,y plane $(x=0)$ could be considered and similar equations (4) and (5) found:

$$2h\left(\frac{\partial z}{\partial y}\right)_{\substack{0\\x=0}} \approx z_2 - z_4 \qquad \ldots (4)$$

$$h^2\left(\frac{\partial^2 z}{\partial y^2}\right) \approx z_2 - 2z_0 + z_4 \qquad \ldots (5)$$

Although this is shown for zero planes, it is readily seen that the equations would hold for any planes parallel to the reference axis. Also, since it will be assumed that these equations are sufficient, the approximately equals sign will be dropped.

To illustrate the use of equations (2) to (5), the equation of interest, which is Laplace's equation, may be approximated at point 0, Figure 12a. For this point:

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)_0 = 0$$

$$h^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)_0 = (z_1 - 2z_0 + z_{-1}) + (z_2 - 2z_0 + z_4) = 0$$

$$z_1 + z_2 + z_3 + z_4 - 4z_0 = 0 \qquad \ldots \quad (6)$$

which gives the condition of interest at point 0, or any arbitrary point in the net.

For obtaining residual and relaxation operators for use with a square net, consider a section of an area in the x,y plane for which a solution to the Laplace equation is desired, where side length of a square is h and the nodes are numbered for reference only, Figure 13a.

From previous discussion it is known from equation (6) that for any point, say point 22

$$\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)_{22} = z_{23} + z_{16} + z_{21} + z_{28} - 4z_{22} = 0$$

and that for each such internal point in the area of interest one such equation may be obtained.

All these equations may be written in what is termed a residual form:

$$\begin{array}{llllll}
z_9 & + z_2 & + z_7 & + z_{14} & - 4z_8 & = R_8 \\
z_{10} & + z_3 & + z_8 & + z_{15} & - 4z_9 & = R_9 \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
z_{30} & + z_{23} & + z_{28} & + z_{35} & - 4z_{29} & = R_{29}
\end{array}$$

where the R values are the residual errors if any set of z values is assumed. In general, if correct values of z for every point were placed in the equations, all residual errors (R values) would be zero, and if values of z which are approximately correct for each point are placed in the equations, the residual errors will be small. However, if values of z which are far from correct are placed in the equations, large residual errors (R values) will result. These statements are usually true. It may also be noted that the set of equations are all of the general form of (6).

These equations are used directly at the beginning of a problem to find the residuals at the nodes by assuming values of z . These residuals are then relaxed and gradually eliminated by "sweeping them over the boundaries" using relaxation operators. The relaxation operators are also independent of nodal points for a region similar to that shown in Figure 13a, and may be derived from the preceding set of equations.

For the relaxation operator, it is desired to obtain some operator which will show the effect on all residuals involved when a change is made in one z value. The change which will be investigated is a change of one unit in a z value. To determine this operator, all equations that influence one typical point, say 22 in Figure 13a, may be written down. Five equations are located.

$$z_{23} + z_{16} + z_{21} + z_{28} - 4z_{22} = R_{22}$$

$$z_{17} + z_{10} + z_{15} + z_{22} - 4z_{16} = R_{16}$$

$$z_{24} + z_{17} + z_{22} + z_{29} - 4z_{23} = R_{23}$$

$$z_{29} + z_{22} + z_{27} + z_{34} - 4z_{28} = R_{28}$$

$$z_{22} + z_{15} + z_{20} + z_{27} - 4z_{21} = R_{21}$$

From these equations the desired operator may be obtained by inspection. Since the value of z_{22} is the value of interest, the effects on the residuals, if this value (z_{22}) is changed by +1, are investigated. It is seen from the equations that the values of:

R_{22} changes by -4 units

R_{16} changes by +1 units

R_{23} changes by +1 units

R_{28} changes by +1 units

R_{21} changes by +1 units

Therefore, for any internal point, the operator may be represented in a form similar to that suggested by Southwell. As this is a general operator, it is shown for a general point notation or instead of using 22, 23, 16, 21, and 28, points 0, 1, 2, 3, and 4 are used. The operator is shown in Figure 13b.

Many other equations and operators are derived in the references. These include conditions for: Nonsquare nets; lines of symmetry (corners and boundaries); soil layer junctions; normal gradient; and interfaces.

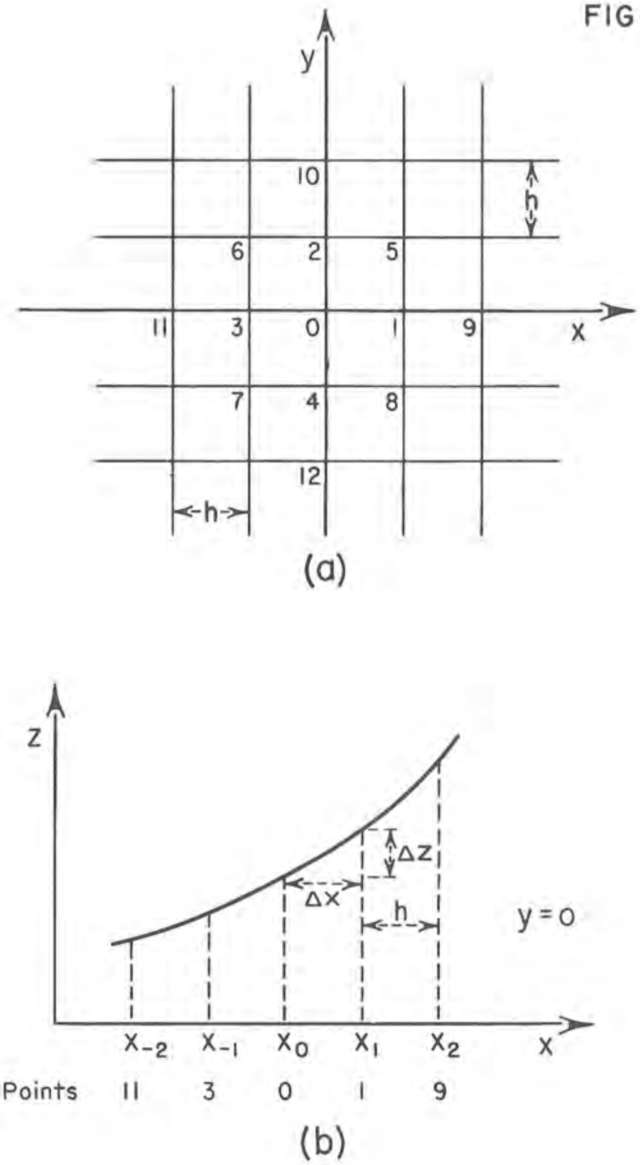
REFERENCES

1. Southwell, R. V., Relaxation Methods in Engineering Science, University Press, Oxford, 1940

2. de G. Allen, D. N., Relaxation Methods, McGraw-Hill Book Company, Inc., 1954

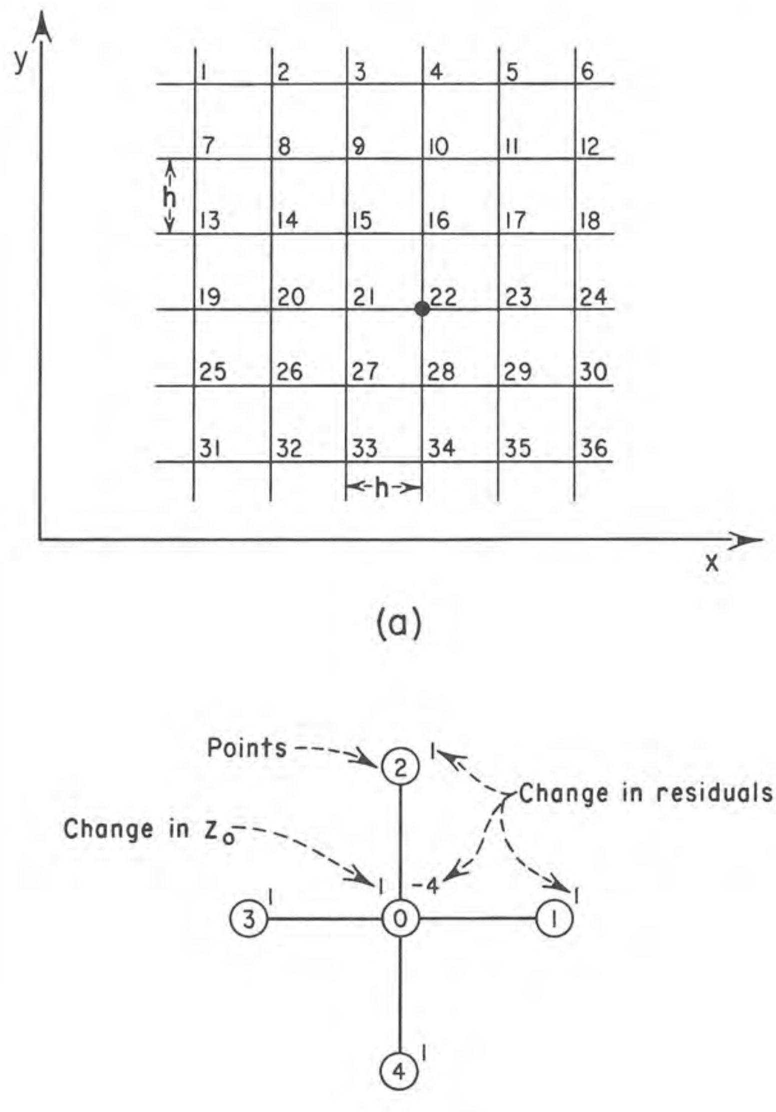
3. Shaw, F. S., Introduction to Relaxation Methods, Dover, 1953

FIGURE 12



SQUARE NET CONDITIONS

FIGURE 13



SQUARE NET AND UNIT RELAXATION OPERATOR