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PRESSURE, ENERGY, AND FLOW CONDITIONS IN CHANNELS WITH HIGH GRADIENTS

by HARALD LAUFFER

Translation from German by
D. P. BARNES, ASSOCIATE ENGINEER

- - -

Denver, Colorado
July 6, 1936

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MEMORANDUM TO CHIEF DESIGNING ENGINEER

Translation from German

SUBJECT: "PRESSURE, ENERGY, AND FLOW CONDITIONS IN CHANNELS WITH HIGH GRADIENTS"

By Harald Lauffer
Assistant at the Technical University
in Graz, Austria

- - -

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Under direction of

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TECHNICAL MEMORANDUM NO.527

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CAUTION

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(Sgd.) J. L. SAVAGE.
Chief Designing Engineer.

PRESSURE, ENERGY, AND FLOW CONDITIONS IN CHANNELS WITH HIGH GRADIENTS

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A Translation of

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PRESSURE, ENERGY, AND FLOW CONDITIONS IN CHANNELS WITH HIGH GRADIENTS

By
Harald Lauffer,
Assistant at the Technical University in Graz.

- - -

1. Notation

c . . . the increment in pressure head resulting from the curvature or change in direction of the stream line.

d . . . the stream thickness or the thickness of the fluid sheet perpendicular to the direction of flow.

g . . . acceleration of gravity.

h_p . . potential energy head.

$h_k = \frac{v^2}{2g}$ . . . the velocity head.

n . . . the perpendicular distance from the water surface.

p . . . hydrostatic pressure.

p_s . . hydrostatic pressure at the bottom.

q . . . flow or discharge per unit width.

$s = \frac{p}{\gamma}$. the pressure head.

t . . . the vertical depth of water.

v . . . the velocity.

z . . . the height above the bed.

$Fr = \frac{v}{\sqrt{gd}}$ . . . the Froude number.

H . . . the energy head above the bed.

K . . . dynamic capacity.

ϕ . . . the angle of inclination with the horizontal,

γ . . . unit weight.

t_{cr}, v_{cr}, q_{cr}, K_{cr}, . . . critical values for horizontal parallel flow.

$d_{q\ max}$, q_{max}, $Fr_{q\ max}$ . . . position of the surface for maximum discharge.

$d_{k\ min}$, K_{min}, $Fr_{k\ min}$ . . . critical values corresponding to the position of the water surface for minimum dynamic capacity.

2. Horizontal Parallel Flow

At any point in a liquid at rest the hydrostatic pressure is equal to the external pressure on the free surface increased by the product of the distance from the surface and the unit weight. If the fluid is in straight-line horizontal motion, there is no change in the pressure condition, and we obtain, with an external pressure $p_0 = 0$, the hydrostatic pressure (fig. 1),

$$p = \gamma(t - z) \tag{1}$$

The total potential energy of a particle of water whose unit weight is 1 may be expressed with reference to the bottom of the channel in terms of the energy of position, z, and the pressure energy, $\frac{p}{\gamma}$:

$$h_p = z + \frac{\gamma(t - z)}{\gamma} = t \tag{2}$$

The potential energy for all particles is therefore constant and equal to the energy of position of a particle on the surface.

If the velocity of a particle is v, then its kinetic energy is equal to the velocity head:

$$h_k = \frac{v^2}{2g}, \tag{3}$$

and the total energy with reference to the bottom of the channel is:

$$H = h_p + h_k = t + \frac{v^2}{2g} \qquad (4)$$

If the energy head for all particles in a cross sect: is plotted vertically above the bottom and if the velocity is c stant throughout this cross section, then all these points will coincide. For these conditions, therefore, we obtain the energ line if for each cross section we plot the velocity head upward from the water surface and connect the points. That applicatic of this expression to the determination of surface profiles and to related problems have shown excellent agreement with actual flow in numerous instances has been noted by Rehbock and other: (see appended list of references). It should not be overlooked, however, that equations 1 and 4 are applicable only under the conditions stated.

Although the introduction of changes of direction i the stream line will entail certain difficulties, the influenc of a uniform slope can be easily derived.

3. Pressure Distribution

Again assuming straight-line parallel flow but wit the stream lines inclined at an angle, ϕ, with the horizontal, we consider a differential element with sides dn and ds (the 1 being considered as unity throughout), the only forces acting the normal direction are the pressure forces and a component the weight (fig. 2). We obtain:

$$p\,ds - \left(p \frac{\partial p}{\partial n} dn\right) ds + \gamma\, dn\, ds \cos\phi = 0,$$

or,

$$\frac{\partial p}{\partial n} = \gamma \cos\phi, \qquad ($$

and finally, for an external pressure, $p_o = 0$ (fig. 3),

$$p = n\,\gamma \cos\phi. \qquad ($$

and the total energy with reference to the bottom of the channel is:

$$H = h_p + h_k = t + \frac{v^2}{2g} \tag{4}$$

If the energy head for all particles in a cross section is plotted vertically above the bottom and if the velocity is constant throughout this cross section, then all these points will coincide. For these conditions, therefore, we obtain the energy line if for each cross section we plot the velocity head upward from the water surface and connect the points. That applications of this expression to the determination of surface profiles and to related problems have shown excellent agreement with actual flow in numerous instances has been noted by Rehbock and others (see appended list of references). It should not be overlooked, however, that equations 1 and 4 are applicable only under the conditions stated.

Although the introduction of changes of direction in the stream line will entail certain difficulties, the influence of a uniform slope can be easily derived.

3. Pressure Distribution

Again assuming straight-line parallel flow but with the stream lines inclined at an angle, ϕ, with the horizontal, if we consider a differential element with sides dn and ds (the breadth being considered as unity throughout), the only forces acting in the normal direction are the pressure forces and a component of the weight (fig. 2). We obtain:

$$p\,ds - \left(p \frac{\partial p}{\partial n} dn\right) ds + \gamma\, dn\, ds \cos\phi = 0,$$

or,

$$\frac{\partial p}{\partial n} = \gamma \cos\phi, \tag{5}$$

and finally, for an external pressure, $p_o = 0$ (fig. 3),

$$p = n\,\gamma \cos\phi. \tag{6}$$

The pressure at the bottom is then

$$p_s = d\,\gamma' \cos\phi = t\,\gamma \cos^2\phi. \qquad (7)$$

The pressure diagram for a straight section therefore remains triangular, although the pressures are smaller for appreciable slopes than those given by equation (1). The graphic determination of the pressure diagram is given in fig. 3.

4. Energy Line

If the general expression for the potential energy of a fluid particle in parallel flow is sought, we obtain from equations (6) and (7)

$$h_p = Z + \frac{p}{\gamma} = t\cos^2\phi + Z\sin^2\phi .$$

For a vertical section (fig. 3), the potential energy of the various particles is no longer constant, but is dependent upon Z, and varies between $h_p = t$ for the upper surface and $h_p = t\cos^2\phi$ for the bottom. If, on the other hand, we consider a normal section, then

$$n = d - \frac{Z}{\cos\phi}$$

and therefore

$$h_p = Z + \frac{p}{\gamma} + \left(d - \frac{Z}{\cos\phi}\right)\cos\phi = d\cos\phi \qquad (8)$$

The energy of all the particles in a normal section is thus seen to be the same.

The total energy of a particle in the normal section whose velocity is v is therefore

$$H = h_p + h_v = d\cos\phi + \frac{v^2}{2g} \qquad (9)$$

with reference to the bottom of the channel.

If the velocity across the normal section is constant, then the energy head for all the particles is the same and the energy line is again obtained by plotting H at each section (fig. 4). It would be relatively simple to plot the velocity heads from points on the surface. When considering friction losses, however, (whenever the energy line is no longer horizontal), this would lead to certain inconsistencies. We are led therefore to the following principal:

The energy line for parallel flow is obtained by plotting the energy head, $H = d \cos \phi + v^2/2g$ above the bottom point of each normal section.

5. Determination of the Position of the Surface and the Maximum Flow

For a given discharge, q, the velocity is

$$v = \frac{q}{d}.$$

When this is introduced into equation (9) we obtain

$$H = d \cos \phi + \frac{q^2}{2gd^2}$$

or

$$d^3 \cos \phi - d^2 H + \frac{q^2}{2g} = 0. \qquad (10)$$

For horizontal flow, we thus obtain the familiar equation for the q curve,

$$t^3 - t^2 H + \frac{q^2}{2g} = 0.$$

For a given energy head and for the critical depth,

$$t_{cr} = \frac{2}{3} H, \qquad (11)$$

the maximum discharge is obtained:

$$q_{cr} = \sqrt{g} \left(\frac{2}{3} H \right)^{\frac{3}{2}} \qquad (12)$$

If equation (10) is divided by H^3, it becomes dimensionless, and, considering equation (12), we obtain:

$$\left(\frac{d}{H}\right)^3 \cos \phi - \left(\frac{d}{H}\right)^2 + \frac{4}{27} \left(\frac{q}{q_{cr}}\right)^2 = 0 \qquad (13)$$

This equation is thus independent of the units of measurement, and as plotted for various slopes in figure 5 is completely general. The solution of the equation for d/H gives three real roots of which one is negative and may therefore be disregarded.

For $q/q_{cr} = 0$ we obtain the intersection of the curve with the d/H axis,

$$\left(\frac{d}{H}\right)_1 = 0 \text{ and } \left(\frac{d}{H}\right)_2 = \frac{1}{\cos \phi} .$$

For a given stream thickness, d, and energy head, H, the discharge according to equation (13) is

$$\frac{q}{q_{cr}} = \sqrt{\frac{27}{4}\left[\left(\frac{d}{H}\right)^2 - \left(\frac{d}{H}\right)^3 \cos \phi\right]} . \qquad (14)$$

The maximum value of the discharge is obtained by differentiating and placing this equation equal to zero:

$$2 \frac{d}{H} - 3 \left(\frac{d}{H}\right)^2 \cos \phi = 0$$

The thickness of the stream is

$$\frac{d_{q\ max}}{H} = \frac{2}{3} \frac{1}{\cos \phi} . \qquad (15)$$

Finally from equations (14 and (15) we obtain:

$$\frac{q_{max}}{q_{cr}} = \frac{1}{\cos \phi} . \qquad (16)$$

The maximum points of the q curves for various slopes (fig. 5) lie on a straight line through the origin whose slope is

$$\frac{d_{q\ max}}{H} : \frac{q_{max}}{q_{cr}} = \frac{2}{3} \tag{17}$$

The calculation of the surface curve in a sloping rectangular channel for a particular discharge q and a particular energy head H (measured from the bottom point of the section to be investigated) is carried out as follows:

First calculate the critical discharge,

$$q_{cr} = \sqrt{g} \left(\frac{2}{3} H\right)^{\frac{3}{2}} ,$$

for the given energy head and for horizontal parallel flow. Next obtain q/q_{cr} and determine the value of d/H either by calculation from equation (13) or graphically from fig. 5. This gives the desired value of the thickness of the stream, d.

In general there will be obtained two values.

$$d_1 > d_{q\ max} \quad , \quad d_2 < d_{q\ max},$$

which coincide for $q = q\ max$: $d_1 = d_2 = d_{q\ max}$:

The value obtained for $q > q_{max}$ is useless since the energy head is then too small to support the necessary discharge.

6. Dynamic Capacity and Resistance to Flow

The dynamic capacity is defined as the sum of the hydrostatic pressure and the transfer of momentum through a cross section in unit time (see reference no. 5). For parallel flow on a slope, ϕ, we have for a normal section (fig. 3),

$$K = \gamma \left(\frac{d^2}{2} \cos \phi + \frac{1}{g} v^2 d\right) \tag{18}$$

For horizontal parallel flow, if

$$d = t_{cr} = \sqrt[3]{\frac{q^2}{g}} , \tag{19}$$

we obtain the limiting value of the dynamic capacity

$$K_{cr} = \sqrt{\frac{3}{2}} \left(\frac{q^2}{g}\right)^{\frac{2}{3}} \tag{20}$$

When equation (18) is divided by t^2_{cr}, and considering equation (20), there is obtained the dimensionless equation for the dynamic capacity.

$$\frac{K}{K_{cr}} = \frac{1}{3} \cos \phi \left(\frac{d}{t_{cr}}\right)^2 + \frac{2}{3} \left(\frac{d}{t_{cr}}\right)^{-1} \tag{21}$$

This equation again is independent of absolute magnitudes, and the dynamic capacity curves shown in fig. 6 for various slopes are completely general. Solving equation (21) for d/t_{cr} yields two useful roots which for a given dynamic capacity can best be determined graphically from fig. 6.

For a particular slope, the maximum value of the dynamic capacity is obtained by differentiating and placing equation (21) equal to zero. This yields, for the thickness of the stream,

$$\frac{d_{k\ min}}{t_{cr}} = (\cos \phi)^{-\frac{1}{3}} \tag{22}$$

The minimum dynamic capacity for a given discharge is therefore

$$\frac{K_{min}}{K_{cr}} = (\cos \phi)^{\frac{1}{3}} \tag{23}$$

The minimum points on the dynamic capacity curve lie on the unit hyperbola

$$\frac{d_{k\ min}}{t_{cr}} \times \frac{K_{min}}{K_{cr}} = 1 \tag{24}$$

For $d = d_{k\,min} = \dfrac{t_{cr}}{\sqrt[3]{\cos \phi}}$, the velocity,

$$v_{k\,min} = \frac{q_{k\,min}}{d_{k\,min}} = \sqrt{g d_{k\,min} \cos \phi}, \qquad (25)$$

is thus equal to the wave velocity in a rectangular channel whose slope is ϕ. If we imagine the same channel in a horizontal position subjected to a gravity field whose acceleration is $g \cos \phi$, then we obtain exactly the same hydrostatic pressure as for the sloping channel in which the acceleration of gravity is g. The wave velocities must therefore be the same, and we obtain, for horizontal channels,

$$v_{cr} = \sqrt{g \cos \phi \; d}.$$

In a sloping channel therefore we obtain, for

$d < d_{k\,min}$, <u>shooting</u> flow, and for
$d > d_{k\,min}$, <u>streaming</u> flow.

The locus of the surface for a minimum dynamic capacity therefore coincides with the boundary between the two types of flow.

7. Position of the Surface for Maximum Discharge, and the Critical Depth

The ratios, d/H and d/t_{cr} can be related through the Froude number,

$$Fr = \frac{v}{\sqrt{gd}}, \qquad (26)$$

which was used by Safranez (see reference no. 9) as the "flow index", R, for the calculation of the hydraulic jump.

From equation (9),

$$v = \sqrt{2g\,(H - d \cos \phi)}.$$

Combining this with equation (26), we obtain

$$Fr = \frac{v}{\sqrt{gd}} = \sqrt{2\left(\frac{H}{d} - \cos\phi\right)}\ .$$

On the other hand, from equation (19)

$$q = t_{cr}^{3/2}\ g^{1/2}\ ,$$

and therefore,

$$Fr = \frac{q}{d\sqrt{gd}} = \left(\frac{t_{cr}}{d}\right)^{\frac{3}{2}}\ .$$

For maximum discharge then, if,

$$\frac{d_{q\ max}}{H} = \frac{2}{3}\ \frac{1}{\cos\phi}$$

the Froude number is

$$Fr_{q\ max} = \sqrt{2\left(\frac{3}{2}\cos\phi - \cos\phi\right)} = \sqrt{\cos\phi},$$

and for the critical condition and minimum dynamic capacity, if

$$\frac{d_{k\ min}}{t_{cr}} = (\cos\phi)^{-\frac{1}{3}},\ \text{we obtain}$$

$$Fr_{k\ min} = \left[(\cos\phi)^{\frac{1}{3}}\right]^{\frac{3}{2}} = \sqrt{\cos\phi}$$

The position of the surface for maximum discharge for parallel flow therefore coincides with the boundary position betwee shooting and streaming flow (fig. 7):

$$Fr_{k\ min} = Fr_{q\ max} = Fr_{crit} = \sqrt{\cos\phi} \qquad (27)$$

The Froude number for this critical flow will therefore differ widely for steep slopes from its value for horizontal parallel flow (Fr = 1).

Under the influence of curvature in the stream line this difference may be considerably magnified, without affecting the validity of the Froude model law (in which the only external forces are assumed to be gravity forces), and without necessitating replacement by any other arbitrary law as suggested by Engel

for the interpretation of the results of experiments on Venturi canals (see references 2 and 3).

8. Increases in Pressure with Curvature of Stream Line

Much effort has recently been expended toward introducing the influence of curvature of the stream lines into elementary methods of calculation. At overflow crests and in the so-called "bucket" the effect of centrifugal force is particularly important (see references 5, 9, and 10).

In order to determine this increase in pressure experimentally, the total pressure is measured, and this is divided into static and dynamic parts. For horizontal parallel flow, this increment in pressure, c, is equal to the difference between the measured pressure head, s, and the vertical distance from the point in question to the surface of the water.

For inclined streamlines the static pressure is no longer equal to the depth below the surface, and we obtain for concentric streamlines (considering equation 6) the increment in pressure head,

$$c = s - n \cos \phi. \qquad (28)$$

The determination of both positive and negative pressure increments due to curvature of the streamlines is indicated in figure 8.

9. Experimental Agreement

Pressure measurements in sections of channels having high gradients have seldom been made in the past. Those that have been completed were for the most part carried out in regions under the influence of considerable curvature. Much valuable data is nevertheless available for checking the fundamental pressure equation no. 7.

Particularly pertinent are the experiments of M. Hasumi at the Vienna Hydraulic Experiment Station. These were concerned with the distribution of pressures along the faces of weirs (see reference 4). The profile of the surface and the pressures along the face were determined for 24 different conditions of flow. Two of these experiments are plotted in figure 9 where the measured pressures are shown for comparison with those calculated from the position of the surface in accordance with equation 7. The average

of the bottom and surface slopes at each normal section was taken as the slope of the stream line in the calculation. Very good agreement between theory and fact was obtained where the curvatures were slight, the discrepancies being well within the limits of experimental accuracy.

10. Conclusion

It has been theoretically and experimentally demonstrated that for parallel flow with high gradients the pressure head in the interior of the liquid is no longer equal to the vertical distance from the surface but is significantly smaller. It follows therefore that the q-line as well as the dynamic capacity are dependent upon the slope. The surface profile corresponding to the minimum dynamic capacity is both the boundary between shooting and streaming flow and the surface profile for maximum discharge. The Froude number for this condition of flow can vary between 0 and 1 depending upon the slope.

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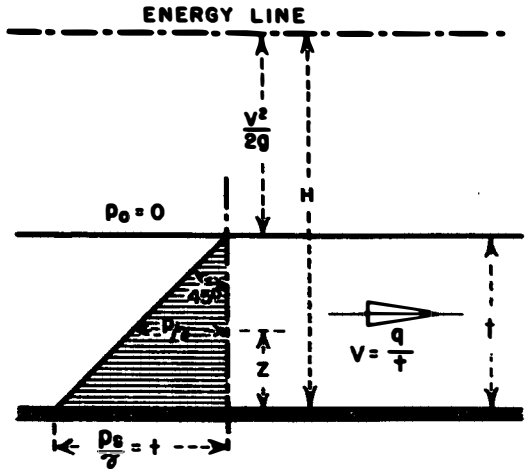

FIG. 1. PRESSURE DISTRIBUTION AND THE ENERGY LINE FOR HORIZONTAL PARALLEL FLOW

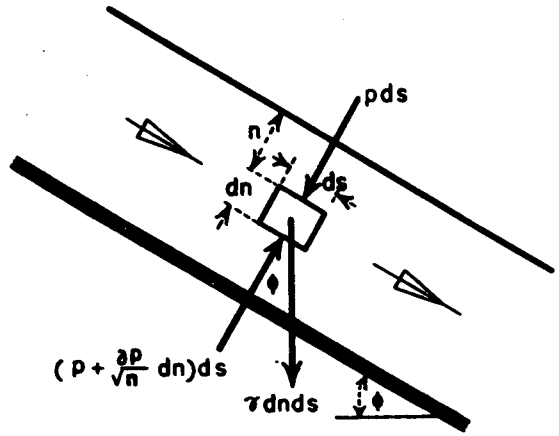

FIG. 2. FORCES ON A NORMAL ELEMENT OF FLOW

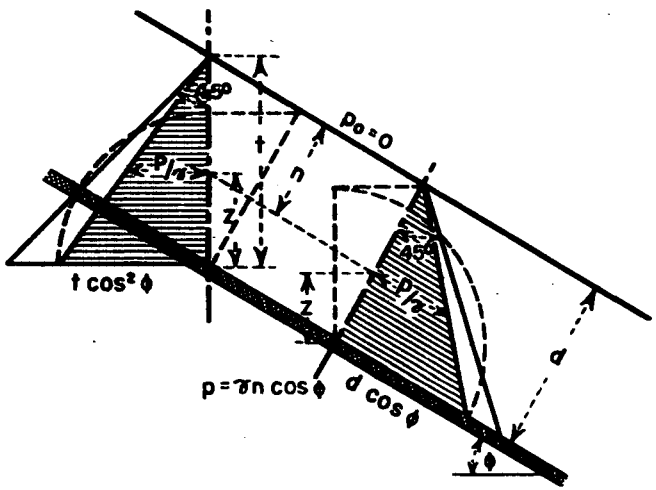

FIG. 3. PRESSURE DISTRIBUTION FOR THE GENERAL CASE OF PARALLEL FLOW

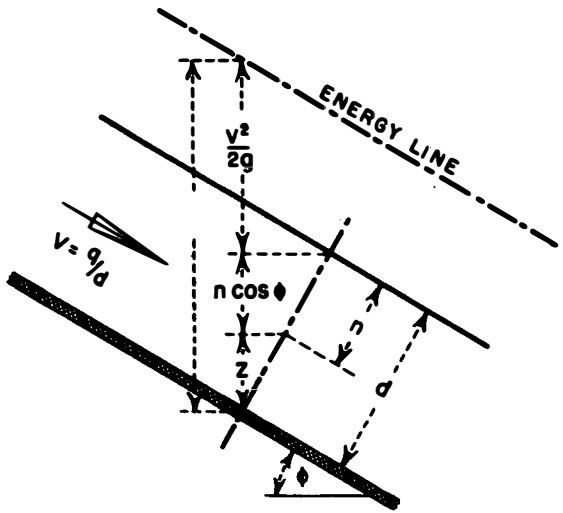

FIG. 4 ENERGY LINE FOR THE GENERAL CASE OF PARALLEL FLOW

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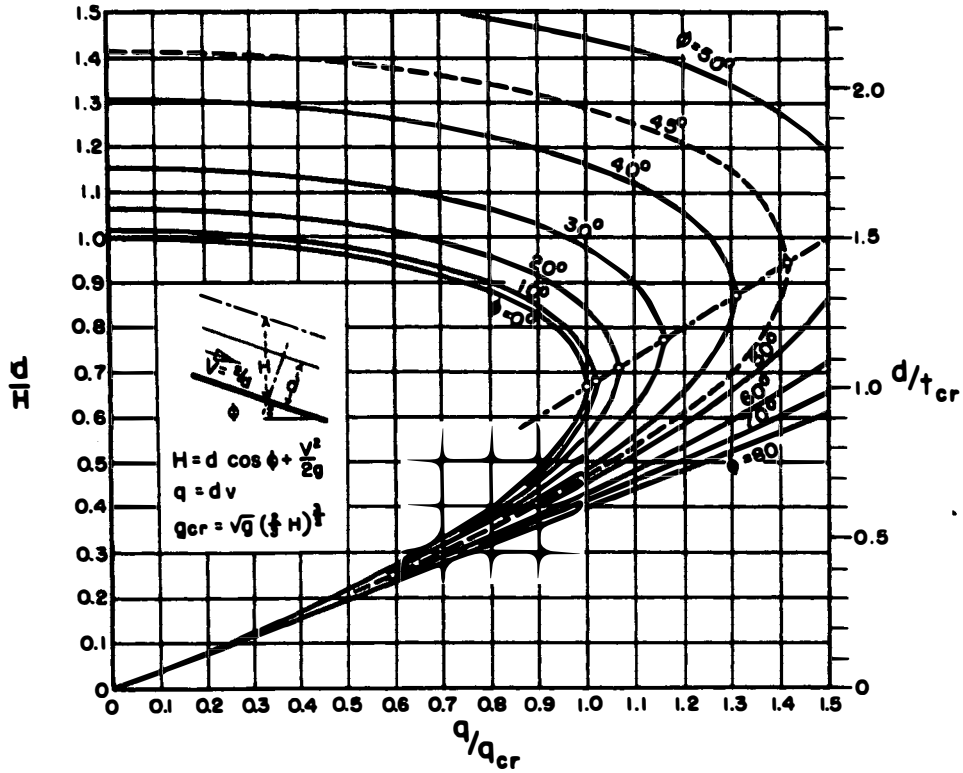

FIG. 5 CURVES OF g FROM EQUATION 13

(VALUES OF $\frac{d}{t_{cr}}$ ARE VALID ONLY FOR HORIZONTAL, PARALLEL FLOW.)

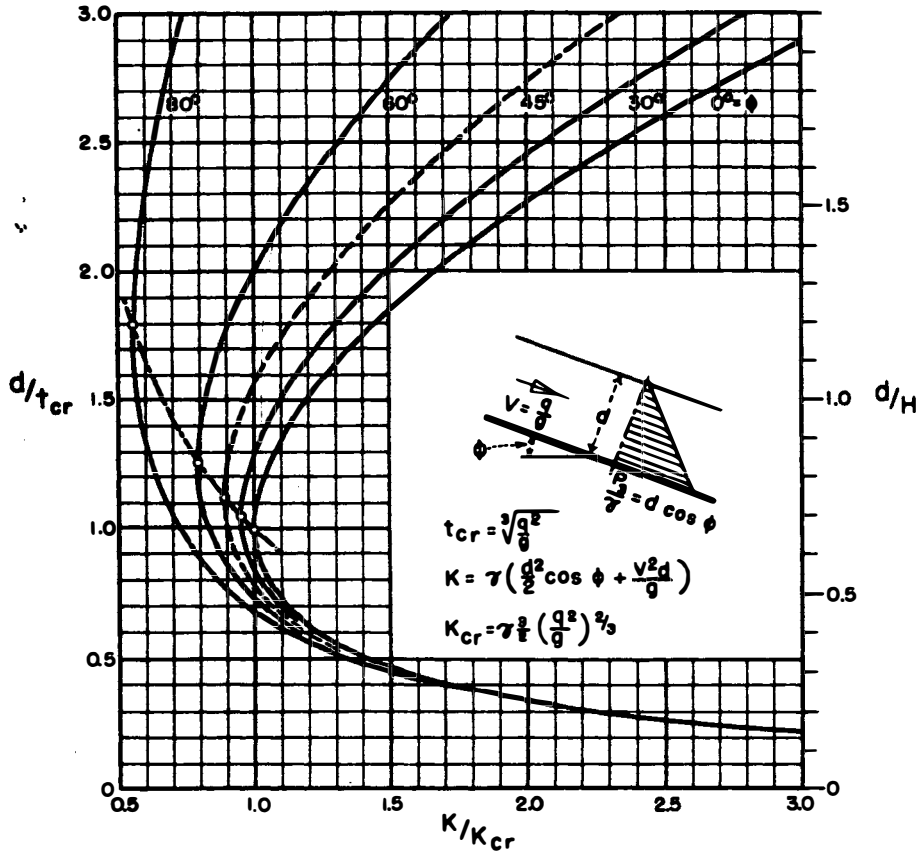

FIG. 6. CURVES OF DYNAMIC CAPACITY FROM EQUATION 21

(VALUES OF $\frac{d}{H}$ ARE VALID ONLY FOR HORIZONTAL PARALLEL FLOW. THE ENERGY HEAD FOR CRITICAL FLOW SHOULD THEREFORE BE USED $H = H_{cr} = \frac{3}{2}\sqrt[3]{\frac{q^2}{g}}$.)

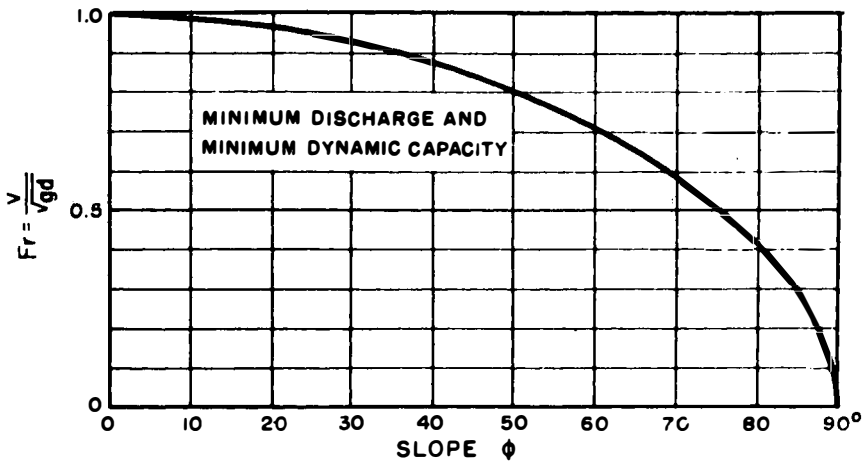

FIG. 7 FROUDE'S NUMBER FOR CHARACTERISTIC POSITIONS OF THE SURFACE

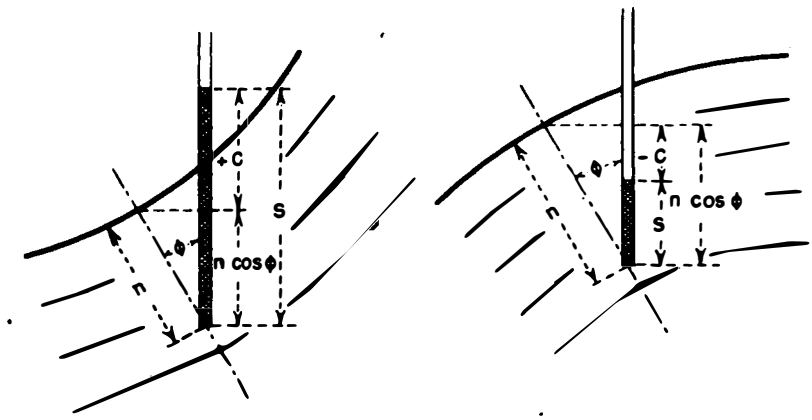

FIG. 8. INCREMENTS IN PRESSURE RESULTING FROM CURVATURE OF FLOW

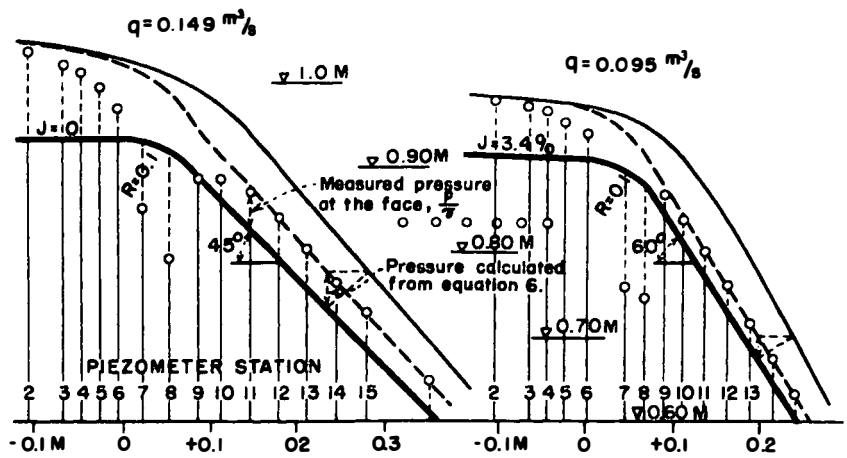

FIG. 9. EXPERIMENTAL AND THEORETICAL DISTRIBUTION OF PRESSURE ON DOWNSTREAM WEIR FACES
(SEE REF. 4 AND FIG'S. 6 AND 12)