UNITED STATES
DEPARTMENT OF THE INTERIOR
BUREAU OF RECLAMATION

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HYDRAULIC LABORATORY REPORT NO. 46

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THE RATE OF LOWERING OF
THE WATERTABLE

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TRANSLATED FROM THE
GERMAN
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Denver, Colorado
1938

The Rate of Ground-Water Lowering

by W. Steinbrenner

A Translation of

Der zeitliche Verlauf einer Grundwasserabsenkung

from

Wasserwirtschaft und Technik

vol. 4, 1937, p. 27.

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The lowering of the ground-water level is due to either the pumping of water for water supply or the unwatering of construction excavations. While in the first case water is being continually removed, in the second case, the draining as well as the unwatering of the excavation during the time of construction will cause the position of the water table to vary with the time. The following study will cover the second case. Others have investigated the problem of the rate of lowering of the water table, namely; Schultze (1), Weber (2) and Kozeny (3). (See references at end of paper).

In order to facilitate the solution of the problem, several simple assumptions are introduced in the following, with the result that the investigation is reduced to an approximation of the actual conditions as given above - First, only small changes in the ground-water level will be considered, which, however, as we shall later show, can be extrapolated. It is permissible to superpose the influence of several wells and hence the problem can be reduced to the case of the single well shown in figure 1. This well, which is discharging at a constant rate, Q, has a filter pipe of radius, r, which extends through a water-bearing strata of thickness, H, down to an impermeable layer. It is further assumed this water-bearing strata is surrounded on all sides by free water; that is, water can percolate freely through a distance $x = L$ to the outer surface of the pipe. Furthermore it is assumed that the water-bearing strata possesses everywhere the same coefficient of permeability, K, and pore volume, n. By the pore volume is meant that part of the voids of the soil which contributes stored water to the flow as the ground-water level sinks. Finally we assume Darcy's law to be valid and, therefore, the filter velocity, v, can be written,

$$v = k \frac{\partial z}{\partial x} \tag{1}$$

in which:

Z = the height of the ground-water level above the impermeable floor; and

x = a distance measured from the axis of the well.

Since we intend to study only small changes with flow that is prevailingly horizontal and steady, the familiar approximate solution of Dupuit holds, namely:

$$H^2 - Z^2 = \frac{Q}{K\pi} \log\left(\frac{L}{x}\right) \tag{2a}$$

This equation represents the situation when the flow is independent of time, that is, when just as much water enters the ground-water basin from the outer regions as is taken from the well. However, when the water table is dropping a part of the discharge is drawn from the ground-water basin itself. Consider a hollow cylinder of radius, x, and thickness, dx, cut out of the water-bearing strata. It is an established fact that the difference between the quantity of water flowing in and out of the surfaces of the cylinder per unit of time is equal to the change in water content of the cylinder during the same time. The quantity of water passing through the cylinder per unit of time is:

$$q = -2\pi k x (H - y) \frac{\partial y}{\partial x} \tag{3a}$$

and the quantity of water stored in the cylinder is:

$$V = 2\pi m x (H - y)\, dx$$

The differential equation for the lowering of the water table is:

$$\frac{\partial q}{\partial x}\, dx = \frac{\partial V}{\partial t}$$

or:

$$-2\pi k \frac{\partial}{\partial x}\left[x (H - y) \frac{\partial y}{\partial x}\right] dx$$

$$2\pi m x \frac{\partial (H - y)}{\partial t}\, dx$$

or

$$\frac{\partial}{\partial x}\left[x (H - y) \frac{\partial y}{\partial x}\right] = \frac{m}{k}\, x\, \frac{\partial y}{\partial t} \tag{4a}$$

Equation (4a) reduces to the previous equation for steady flow (2a). This equation is not linear but for small changes in the water-table level it can be replaced by equation (4) which is linear. For small depressions of the water table we have approximately:

$$H - y \doteq H$$

and instead of (3a) the discharge becomes:

$$q = -2\pi k x H \frac{\partial y}{\partial x} \tag{3}$$

and in place of (4a) we have:

$$\frac{\partial}{\partial x}\left[Hx\frac{\partial y}{\partial x}\right] = \frac{m}{K}\, x\, \frac{\partial y}{\partial t}$$

or, $$\frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\frac{\partial y}{\partial x} = \frac{m}{HK}\frac{\partial y}{\partial t} \qquad (4)$$

Placing $\frac{\partial y}{\partial t} = 0$ in (4), we obtain the differential equation for steady flow, of which the solution has the form:

$$y = d_1 \log x + d_2$$

With the corresponding choice of constants of integration this solution may be written:

$$y = \frac{Q}{2\pi KH}\log\left(\frac{L}{x}\right) \qquad (2)$$

Equation (2) can be obtained directly from (2a), thus from (2a):

$$\frac{Q}{\pi K}\log\left(\frac{L}{x}\right) = H^2 - z^2 = 2Hy - y^2$$

which by neglecting the y^2- term gives equation (2). Since we wish to arrive at the lowest position of the water table ($t = t_0$) corresponding to steady flow, starting from the horizontal position of the water table ($t = 0$) and with a constant Q, it is our problem to find a function $y = f(x, t)$ which satisfies differential equation (4) and also the following boundary conditions:

Boundary conditions:

$$q = Q \text{ at } x = r \qquad (5a)$$

$$y = 0 \text{ at } x = L \qquad (5b)$$

Time conditions;

$$y = o \text{ when } t = o \tag{5c}$$

$$y = \frac{Q}{2\pi KH} \log\left(\frac{L}{x}\right) \text{ when } t = t_c \tag{5d}$$

Since equation (4) is linear, the theorem of superposition is applicable. Thus the sum of the particular solutions is a solution of the differential equation. We solve the problem by separating it in simple way and write:

$$y = y_1 + y_2 \qquad\qquad q = q_1 + q_2$$

in which y_1 and y_2 must satisfy differential equation (4). Functions q_1 and q_2 are given by the following relations:

$$q_1 = -2\pi KH \times \frac{\partial y_1}{\partial x} \qquad q_2 = -2\pi KH \times \frac{\partial y_2}{\partial x}$$

We shall use solution (2) for y_1. This corresponds to the final steady flow, for which the discharge of water at any distance x, thus also for x = r, is constant and equal to the quantity of water pumped, Q, and which satisfies the condition for y = o at x = L. In order to satisfy the boundary conditions (5a to d) we must superpose on the solution y_1, the second solution y_2, which satisfies the following conditions:

$$q_2 = o \quad \text{at } x = r \tag{6a}$$

Place conditions

$$y_2 = o \quad \text{at } x = L \tag{6b}$$

Time conditions $\quad y_2 = -\frac{Q}{2\pi KH} \log\left(\frac{L}{x}\right) \text{ when } t = 0 \quad$ (6c)

$$y_2 = o \quad \text{when } t = t_c \tag{6d}$$

In order to obtain a particular integral of (4) we place $y = XT$ in which X is a function of x only and T of t only. Thus we find:

$$\frac{d^2X}{dx^2}(T) + \frac{1}{x}\frac{dX}{dx}(T) = \frac{n}{KH}(X)\frac{dT}{dt}$$

or

$$\frac{1}{X}\left[\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx}\right] = \frac{n}{KH}\frac{1}{T}\frac{dT}{dt} \qquad (7)$$

By placing both sides of (7) equal to an arbitrary constant, $-\left(\frac{\alpha}{L}\right)^2$ we obtain the two ordinary differential equations:

$$\frac{1}{T}\frac{dT}{dt} = -\frac{HK}{nL^2}\alpha^2 \qquad (8)$$

and

$$\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} + \left(\frac{\alpha^2}{L^2}\right)X = 0 \qquad (9)$$

or

$$\frac{d^2X}{d\left(\alpha\frac{x}{L}\right)^2} + \frac{1}{\alpha\frac{x}{L}}\frac{dX}{d\left(\alpha\frac{x}{L}\right)} + X = 0$$

For an arbitrary value of α it follows from (8) that:

$$T = e^{-\alpha^2\tau}$$

in which:

$$\tau = \frac{HKt}{nL^2}$$

Since equation (9) is satisfied by a Bessel function of zero order, we may write the desired second part of the solution, y_2 as an infinite series in terms of Bessel's function, with the provisional but undetermined coefficient, A, thus:

$$y_2 = \sum^{\alpha} A\, J_0 \left(\alpha \frac{x}{L} \right) e^{-\alpha^2 \tau}$$

Then:

$$q_2 = 2\pi K H \frac{x}{L} \sum^{\alpha} A\, \alpha\, J_1 \left(\alpha \frac{x}{L} \right) e^{-\alpha^2 \tau}$$

and permit α which is introduced for the positive roots of the equation $Jo(x) = o$ to run throughout the series. In these equations Jo and J_1 are Bessel functions of the zero and first order, respectively. The relation between these two functions is given by this expression:

$$\frac{d\, J_0\left(\alpha \frac{x}{L}\right)}{dx} = -\frac{\alpha}{L} J_1\left(\alpha \frac{x}{L}\right)$$

We can easily satisfy ourselves that y_2 according to the above equation fulfills the boundary conditions (6b) and (6d).

For $x = L$, $y_2 = o$, because α is a root of Jo. For $t = t_c = \infty$, $y_2 = o$ because the factor $e^{-\alpha^2 \tau}$ vanishes. Further, the radius, r, of the well is very small and therefore for small values of α:

$$\frac{\alpha x}{L} = 0$$

and with this:

$$J_1\left(\frac{x}{L}\right) = o$$

At $x = r$, only the terms with large values of α contribute essentially to the discharge which, however, decreases very rapidly with the time, on account of the factor $e^{-\alpha^2 \tau}$. The [illegible] (8c) is therefore not satisfied for very small [illegible]. This will be treated in an approximate way later.

[illegible] to satisfy boundary condition (8c) we choose the coeffi- [illegible] now undetermined, so that for

$$[illegible] = 0,\ T_2 = T_1 = \frac{Q}{2\pi KH} \log\left(\frac{x}{L}\right) = f(x)$$

This condition is satisfied between the limits $\frac{x}{L} = 0$ and [illegible], if the coefficient A corresponding to a given root, α, is [illegible]

$$A = \frac{2}{J_1(\alpha)^2} \int_0^1 f(x)\, J_0\left(\alpha \frac{x}{L}\right) \frac{x}{L}\, d\left(\frac{x}{L}\right)$$

[illegible]

$$\alpha \frac{x}{L} = \xi \quad \text{and} \quad f(x) = \frac{Q}{2\pi KH} \log\left(\frac{x}{L}\right)$$

$$A = \frac{Q}{\pi KH\, \alpha^2 J_1(\alpha)^2} \int_0^\alpha \log\left(\frac{\xi}{\alpha}\right) J_0(\xi)\, \xi\, d\xi$$

$$\int_0^\alpha \log\left(\frac{\xi}{\alpha}\right) J_0(\xi)\, \xi\, d\xi$$

$$= \Big[\xi J_1(\xi) \log(\xi) + J_0(\xi) -$$

$$- \xi J_1(\xi) \log \alpha \Big]_{\xi=0}^{\xi=\alpha} = -1$$

Thus after the integration and the substitution of the limits we obtain:

$$A = -\frac{Q}{\pi KH \alpha^2 J_1(\alpha)^2}$$

and for y_2:

$$y_2 = -\frac{Q}{\pi K H} \sum \frac{J_0(\alpha \frac{x}{L})}{\alpha^2 J_1(\alpha)^2} e^{-\alpha^2 \tau} \tag{10}$$

The lowering, y, according to (2) and (10) is thus:

$$y = y_1 + y_2 = \frac{Q}{2\pi K H}\left[\log\left(\frac{L}{x}\right) - 2\sum^{\alpha} \frac{J_0(\alpha \frac{x}{L})}{\alpha^2 J_1(\alpha)^2} e^{-\alpha^2 \tau}\right] \tag{11}$$

From (11) we obtain:

$$\frac{\partial y}{\partial x} = \frac{Q}{2\pi K H}\left[-\frac{1}{x} + \frac{2}{L}\sum^{\alpha} \frac{J_1(\alpha \frac{x}{L})}{\alpha J_1(\alpha)^2} e^{-\alpha^2 \tau}\right]$$

from which the equation for the discharge is obtained, thus:

$$\frac{q}{Q} = 1 - 2\frac{x}{L} \sum^{\alpha} \frac{J_1(\alpha \frac{x}{L})}{\alpha J_1(\alpha)^2} e^{-\alpha^2 \tau} \tag{12}$$

Series (11) and (12) converge well for large values of τ, that is, if the influence of the boundary of the zone of influence is already appreciable.

We shall now greatly decrease the radius L of the zone of influence, that is, investigate the functions, y, and, q, for small values of

$$\tau = \frac{HKt}{\mu L^2}$$

and point out next, that for large values of α the following relations are valid:

$$\alpha = a\pi(p - 0.25) \doteq \pi(p - 0.25)$$

in which:

$p = 1, 2, 3, 4, \ldots\ldots\ldots\ldots$

Then:

$$J_1(\alpha) = \frac{1}{\sqrt{b}} \frac{\cos\left[\pi(p-0.25)-\frac{3\pi}{4}\right]}{\sqrt{\frac{1}{2}\pi\alpha}}$$

or
$$\alpha J_1(\alpha)^2 = \frac{1}{b}\cdot\frac{2}{\pi} = \frac{2}{\pi}$$

As the following table shows, factors a and b are to be used only in the first four terms of the series. For all higher terms, b = a = 1 is valid, without any appreciable error.

TABLE 1

p	α	$J_1(\alpha)$	$b = \frac{2}{\pi\alpha J_1(\alpha)^2}$	$a = \frac{\alpha}{p-0.25}$	$\frac{b}{a}$
1	2.405	+0.5191	0.982	1.020	0.964
2	5.520	-0.3403	0.995	1.004	0.991
3	8.654	+0.2715	0.997	1.002	0.996
4	11.792	-0.2325	0.999	1.001	0.998
5	14.931	+0.2065	0.999	1.001	0.999
6	18.071	-0.1877	1.000	1.000	1.000

Since series 11 converges very rapidly for small values of τ, we replace it by an integral and obtain by considering the above relations for α and $\alpha J_1(\alpha)^2$ the following equation in place of (11):

$$v\frac{kH}{Q} = \frac{1}{4\pi}\left[\log\left(\frac{L}{x}\right)^2 - 2\int_{\rho=\frac{1}{2}}^{\rho=\infty} \frac{b}{a}\,\frac{J_0(\alpha\frac{x}{L})}{\alpha}\, e^{-\alpha^2\tau}\, d\alpha\right] \qquad (11b)$$

Furthermore we shall replace the Bessel function by an infinite power series [5] thus:

$$J_0\left(\alpha\frac{x}{L}\right) = 1 - \frac{\left(\alpha\frac{x}{2L}\right)^2}{(1!)^2} + \frac{\left(\alpha\frac{x}{2L}\right)^4}{(2!)^2} - \frac{\left(\alpha\frac{x}{2L}\right)^6}{(3!)^2} + \cdots$$

and obtain for the required integral the following:

$$2\int_{\rho=\frac{1}{2}}^{\rho=\infty} \frac{b}{a}\,\frac{J_0(\alpha\frac{x}{L})}{\alpha}\, e^{-\alpha^2\tau}\, d\alpha =$$

$$A_1\underbrace{\int_{\rho=\frac{1}{2}}^{\rho=\infty} \frac{b}{a}\,\frac{e^{-\alpha^2\tau}}{\alpha^2\tau}\, d(\alpha^2\tau)}_{\mathrm{I}} - A_2\underbrace{\int_{\rho=\frac{1}{2}}^{\rho=\infty} e^{-\alpha^2\tau}\, d(\alpha^2\tau)}_{\mathrm{II}}$$

$$+ A_3\underbrace{\int_{\rho=\frac{1}{2}}^{\rho=\infty} (\alpha^2\tau)\, e^{-\alpha^2\tau}\, d(\alpha^2\tau)}_{\mathrm{III}} - A_4\underbrace{\int_{\rho=\frac{1}{2}}^{\rho=\infty} (\alpha^2\tau)^2 e^{-\alpha^2\tau}\, d(\alpha^2\tau)}_{\mathrm{IV}} + \cdots$$

in which the constants $A_1, A_2, A_3, \ldots\ldots$ are given by the following expressions:

$$A_1 = 1 \;, \quad A_2 = \frac{\left(\frac{x}{2L}\right)^2}{\tau\,(1!)^2} = \frac{\varepsilon}{4(1!)^2}$$

where $A_3 = \dfrac{\left(\frac{x}{2L}\right)^4}{\tau^2(2!)^2} = \dfrac{\varepsilon^2}{4^2(2!)^2}$; $A_4 = \dfrac{\left(\frac{x}{2L}\right)^6}{\tau^3(3!)^2} = \dfrac{\varepsilon^3}{4^3(3!)^2}$

$$\frac{x^2}{L^2\tau} = \frac{S x^2}{Hkt} = \varepsilon$$

Several difficulties appear for the first time in the above integrals, because in these the small α range possesses a large influence and hence the approximation b/a = 1 is not justified for small values of a. We resort, therefore, to a series and integrate beginning with p = 4.5.

$$\int_{p=\frac{1}{2}}^{p=\infty} \frac{b}{a} \frac{e^{-\alpha^2\tau}}{\alpha^2\tau} d(\alpha^2\tau) = 4 \sum_{\alpha_1}^{\alpha_4} \frac{1}{\alpha^2 J_1(\alpha)^2} +$$

$$\int_{p=4.5}^{p=\infty} \frac{e^{-\alpha^2\tau}}{\alpha^2\tau} d(\alpha^2)\tau = 4.953 - Ei\left[-(4.25\pi)^2\tau\right]$$

$$= \log(141.6) - \log\left[\gamma(4.25\pi)^2\tau\right]$$

in which $\gamma = 1.7811$.

Since the value of τ is very small, we can put in the above relation

$$e^{-\alpha^2\tau} \doteq 1 \qquad \text{(range from } \alpha_1 \text{ to } \alpha_4\text{)}$$

and put:

$$Ei\left[-(4.25\pi)^2\tau\right] \doteq \log\left[\gamma(4.25\pi)^2\tau\right]$$

If we add equation (11b) to the expression above having the logarithmic term, we obtain

$$\log\left(\frac{L^2}{\chi^2}\right) - \int_{p=\frac{1}{2}}^{p=\infty} \frac{b}{a} \frac{e^{-\alpha^2\tau}}{\alpha^2\tau} e^{-\alpha^2\tau} d(\alpha^2\tau)$$

$$= \log\left(\frac{\left(\frac{L^2}{\chi^2}\right)\gamma(4.25\pi)^2\tau}{141.6}\right) = \log\left(\frac{1.5^2}{\varepsilon}\right)$$

The remaining integrals, II, III, IV, V . . . . . are easy to compute and give, after substituting the limits:

II = 0! = 1, III = 1!, IV = 2!, V = 3! VI = 4!

We therefore have for the lowering of the water table the expression:

$$y\frac{kH}{Q} = \frac{1}{4\pi}\left[\log\frac{1.5^2}{\varepsilon} + \frac{\varepsilon}{4.1!} - \frac{\varepsilon^2}{4^2\,2.2!} + \frac{\varepsilon^3}{4^3\,3.3!} - \frac{\varepsilon^4}{4^4\,4.4!} + \cdots\cdots\right] \quad (11a)$$

From which the equation for the discharge is easily determined, thus:

$$1-\frac{q}{Q} = \frac{1}{4.1!}\varepsilon - \frac{1}{4^2 2!}\varepsilon^2 + \frac{1}{4^3 3!}\varepsilon^3 - \frac{1}{4^4 4!}\varepsilon^4 \cdots\cdots \quad (12a)$$

The values in table II for $y\,\frac{HK}{Q}$ and $\frac{q}{Q}$ are computed from equations (11a) and (12a).

TABLE II

ε	$y\frac{HK}{q}$	$\frac{q}{Q}$	ε	$y\frac{HK}{q}$	$\frac{q}{Q}$
25	0.0065	0.012	0.30	0.1661	0.928
10	0.0081	0.082	0.17	0.2068	0.958
6	0.0261	0.225	0.10	0.2497	0.975
3	0.0372	0.472	0.056	0.2949	0.986
1.7	0.0528	0.653	0.032	0.3390	0.992
1.0	0.0852	0.778	0.018	0.3846	0.995
0.6	0.1164	0.860	0.010	0.4312	0.997

As already mentioned, the discharge during the lowering of the water table is not constant, but does attain the value Q after a very short interval, t_1. Taking the last value in table II, namely:

$$\mathcal{E} = 0.010 \div y \frac{HK}{Q} = 0.4312 \div \frac{q}{Q} = 0.997$$

as a basis, we find:

$$t_1 = 100 \frac{nr^2}{HK} \qquad (13)$$

Presentation of the Results

As already mentioned, series (11) and (12) converge only for large values of τ. In this region the lowering y and the discharge q were computed with the aid of the Function Tables by Jahnke and Emde (5). The results are given in figures 2 and 3. Figure 2 shows the discharge of water as a function of the time which is plotted as abscissae on a logarithmic scale. The curve for $x/L = 1$ is especially important, because it shows the influence of the size of the zone influence. We see that for a value of $\tau < 0.04$ no appreciable quantity of water enters the boundary of the zone of influence, thus for $\tau < 0.04$, it can be considered to be unlimited. For $\tau = 1$ on the contrary, the entire, Q enters at the boundary of the zone of influence; in other words, constant conditions almost obtain at all points.

Figure 3 shows the position of the water table for various values of τ. $\frac{x}{L}$ is plotted as abscissae on a logarithmic scale. According to equation (11a) we have:-

$$y = f(\varepsilon), \quad \varepsilon = \frac{nx^2}{HKt}$$

or

$$x = \sqrt{t \cdot F(y)}$$

$$\log x = \tfrac{1}{2} \log (t) + \log [F(y)]$$

therefore

$$\frac{\partial}{\partial t} (\log x) = \frac{1}{2t}$$

Therefore in figure 3, for the valid range of equation (11a) the horizontal distance between any two adjacent curves is constant and we can represent the successive curves in a simple manner by the parallel displacement of any curve in the x- direction. From equations (11a) and (12a) we obtain for small values of ε the following limiting value:

$$y \frac{KH}{Q} = \frac{1}{4\pi} \log \left(\frac{1.5^2}{\varepsilon}\right) \qquad (14)$$

$$\frac{q}{Q} = 1 \qquad (15)$$

From equation (14) we have:

$$\frac{\partial y}{\partial t} = \frac{Q}{4\pi K H t}$$

For this range (ε small), it is evident on account of the straight-line tendency of the curves in figure 3, that parallel lowering of the ground-water level (parallel displacement in the y-direction) is valid and is confirmed by observations. Thus we agree with Weber [2], who was correct in remarking that these parallel displacements begin to appear as the lowering proceeds.

Finally, we shall investigate closely the significance of the function y_2. As we see from boundary conditions (6a to 6d), negative values of y_2 give us the position of the ground-water level if we stop pumping completely after attaining constant conditions. Thus the initial position is given by the discharge Q as previously found and is to be computed starting from the time of stopping the pump. For determining the rise of the level we write:

$$y_3 = -y_2 = y_1 - y$$

or

$$y_3 \frac{4\pi KH}{Q} = \log\left(\frac{L^2}{x^2}\right) - 4\pi y \frac{KH}{Q} \qquad (16)$$

Within the range of small τ (that is $\tau < 0.04$) the value of $y \frac{KH}{Q}$ in table II can be used. If we introduce for $y \frac{KH}{Q}$ in (16) its value from (11a) we obtain:

$$y_3 \frac{4\pi KH}{Q} = \log \frac{1}{2.25\tau} - \frac{\varepsilon}{4.1.\ 1!} + \frac{\varepsilon^2}{4^2.3.3\ !} - \frac{\varepsilon^3}{4^3.3.3!} + \ldots \qquad (16a)$$

The rising of the water table is shown in figure 4.

The Zone of Lowering

We can write equation (14) in the following form, noting that $\varepsilon = \frac{mx^2}{HKt}$:

$$y = \frac{Q}{2\pi KH} \log\left(\frac{1.5\sqrt{\frac{HKt}{m}}}{x}\right) = \frac{Q}{2\pi KH} \log\left(\frac{R}{x}\right) \qquad (17)$$

$$R = C\sqrt{\frac{HKt}{m}} = 1.5\sqrt{\frac{HKt}{m}} \qquad (18)$$

Equation (17) is approximately valid in the range for which the discharge q is practically the same as the demand Q and differs from equation (2) only in that instead of a constant, L, a function R, of the time enters. R is called the zone of the lowering. For a constant discharge we can also consider the variation in the depth of the ground-water flow and write the following expression analagous to equation (2):

$$H^2 - Z^2 = \frac{Q}{K\pi} \log\left(\frac{R}{x}\right) \tag{17a}$$

Equations (14), (17), and (17a) do not satisfy differential equations (4) and (4a) and thus are not mathematically correct. The error in the useable range of the equations, however, is not very large, since for small values of ε in (4) and (4a) the term with $\frac{\partial y}{\partial t}$ is very small compared to the other terms containing $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial x^2}$.

Approximate formulas for the Zone by Schultze [1], Weber and Kozeny exist as well as an empirical formula by Sichardt (7). According to Weber:

$$R = 3\sqrt{\frac{HKt}{n}}$$

and

$$y = H - \sqrt{H^2 - \frac{Q}{K\pi}\left(\log\frac{R}{x} - \frac{1}{2} + \frac{x^2}{2R^2}\right)}$$

Kozeny [3], on the other hand, obtained the following equations:

$$R = \sqrt{\frac{12t}{m}}\sqrt{\frac{Qk}{\pi}} = \sqrt{\frac{12}{H\sqrt{\pi}}}\sqrt{\frac{Q}{k}}\sqrt{\frac{Hkt}{m}}$$

while he retained Weber's equation for the lowering of the water table.

It should be stated here that the concept of a zone contains in itself an assumption which is only an approximation, and that the coefficient $c = 1.5$ in equation (8) is valid only for the curves for the profile of the water table computed from (17) and (17a). If we accept that at a distance $x = R$ the discharge amounts to only a certain fraction of the quantity of water drawn from the well; for example, $q = 0.012\ Q$, then we obtain for the above coefficient, according to table II, with $\varepsilon = \frac{nx^2}{HKt} = 16$, the value $c = \sqrt{\varepsilon} = 4$, while the limiting value of $\tau = 0.04$ mentioned several times previously, corresponds to a value of $c = 5$. Furthermore we can define the zone in terms of a definite lowering $y = \frac{H}{m}$ for $x = R$. Since according to (11a) the lowering is directly proportional to the discharge of the well and since ordinarily we speak of a lowering only where it can be measured, the zone width according to this definition is also a function of the discharge of the well.

We obtain from table II:

$$y \frac{H\ k}{Q} = f\ (\varepsilon) \text{ or } \varepsilon = F\left(\frac{HK}{yQ}\right) = F\ (\eta)$$

from which the following expression for the zone width results:

$$F = \sqrt{F(\eta)} \; \sqrt{\frac{HKt}{n}} \tag{19}$$

The coefficient $c = \sqrt{F(\eta)}$ in equation (19) is a function of $\eta = \frac{H^2 K}{M\,Q}$ and can be computed by means of table II. For the special case $Q = 0$, we obtain from Kozeny's equation, as well as from equation (19), the zone width $F = 0$, that is, there is no lowering. The zone width according to equation (16) on the contrary is independent of the discharge and has more the significance of an auxiliary expression for the calculation. The influence of the discharge of the well appears first in this case in the amount of lowering.

Since in this paper as well as in studies of Weber and Kozeny simple assumptions were made, we wish to compare the results by means of numerical examples.

<u>Example</u>

Suppose the following quantities are given for the water table and the well:

$H = 10m$

$L = 600m$

$n = 0.3$

$K = 0.001 \; m/s = 86.4 \; m/day$

$Q = 0.020 \; m^3/s = 1728 \; m^3/day$

According to equation (13) the quantity taken from the well for values of $t > t_1$, is approximately constant and we obtain considering the radius, r, of the well to be 0.5 meter:

$$t_1 = 100 \frac{n\, r^2}{HK} = 12.5 \text{ minutes.}$$

The ground water can be considered limitless up to $\tau = 0.04$ or up to

$$t_2 = 5 \text{ days.}$$

Steady conditions are reached at $t_3 = 125$ days. For a time $t = 4$ days we obtain for the zone width the following values according to:

Equation (18)	$R = 161$ m.
Weber	$R = 322$ m.
Kozeny	$R = 187$ m.
Equation (19)	$R = 192$ m.

(for $m = 200$, $Q = 0.02\ m^3/s$).

Using the same time ($t = 4$ days) we obtain according to table II, equations (17) and (17a) as well as according to Weber and Kozeny the values for the lowering y in table III which are also shown graphically in figure 5.

According to table III we can divide the zone of influence into two regions, namely - in an inner region from $x = r$ to $x = 44$ m and an outer region from $x = 44$ m to $x = L$. Outwardly the lowerings are small in comparison with the depth Z of the ground-water flow, so that we are entitled to calculate y from table II or equation (11a). For the inner region, on the other hand, the discharge is approximately constant ($\frac{q}{Q} = 0.958$ to 1.00) and we can, therefore, compute the lowering by considering the

TABLE III

ε					0.032	0.056	0.17	0.60	1.7	6	16
X	0.1	1.0	3.0	10	19.200	25.399	44.354	63.138	139.943	248.906	429.385
y from equation (17)	-	-	-	-	0.68	0.59	0.41	0.21	0.04	-	-
y from table II	-	-	-	-	0.68	0.59	0.42	0.23	0.11	0.02	0.001
y from equation (17a)	2.72	1.77	1.36	0.93	0.70	0.61	0.42	0.21	0.05	-	-
q/Q from table II	-	-	-	-	0.99	0.99	0.96	0.86	0.65	0.23	0.01
y according to Weber	2.80	1.85	1.43	1.00	0.77	0.67	0.49	0.29	0.14	0.01	-
y according to Kozeny	2.57	1.64	1.23	0.81	0.56	0.49	0.3.	0.13	0.02	-	-

variation of Z in equation (17a). The value for the boundary itself calculated from both equations are in approximate agreement. Finally, we point out that the values for the lowering determined from table II and equation (17a) lie between the corresponding values computed from Weber's and Kozeny's formulas and their computation is moreover extremely simple, so that this method of calculation in the opinion of the author, is suitable for practical purposes.

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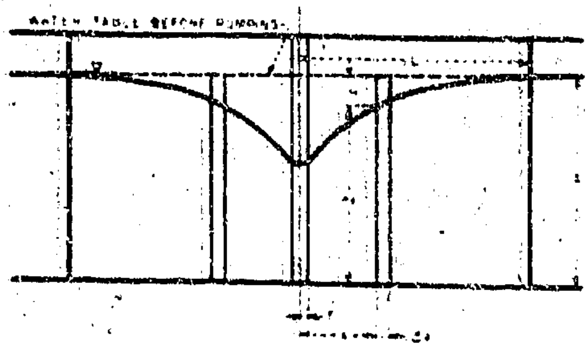

FIGURE 1

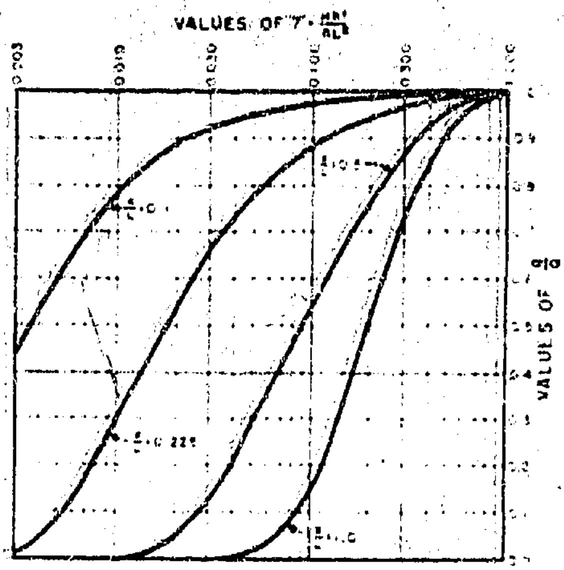

FIGURE 2

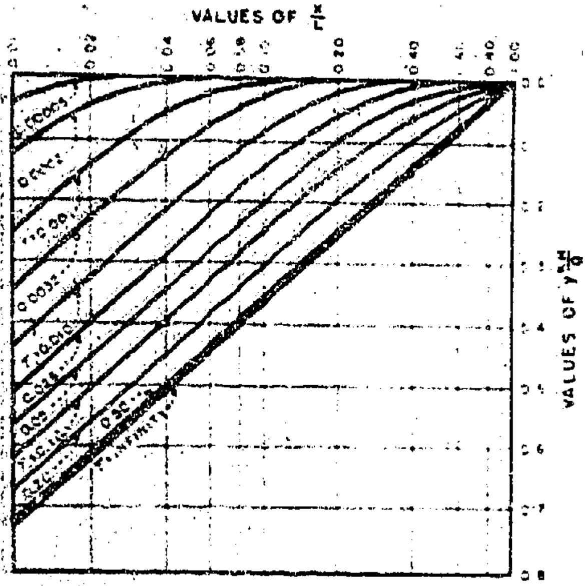

FIGURE 3

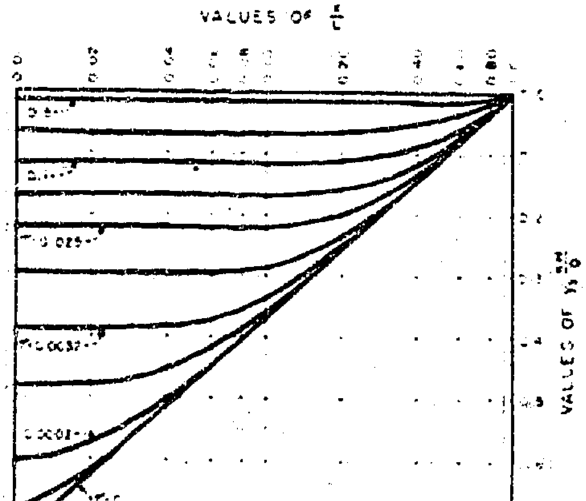

FIGURE 4

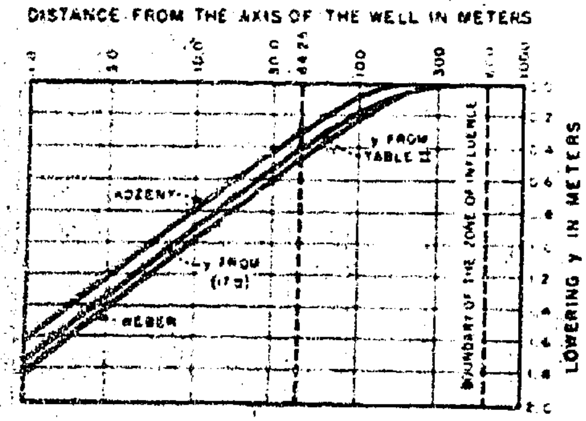

FIGURE 5

K-D-2197

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