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HYD 292

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Denver, Colorado

March 25, 1949

Memorandum

To: E. W. Lane

From: R. E. Glover

Subject: The stable channel problem for coarse material.

References

1. Reference is made to your memorandum of January 21, 1949, on the above subject and to the references quoted therein. Reference is also made to my memorandum of February 1, 1949, on the above subject and to Mr. Swain's memorandum of February 15, 1949, to Mr. E. W. Lane on "Check of mathematical work."

Summary

2. In the writer's memorandum of February 1, 1949, the shape of a stable channel is derived on the basis that the tendency to motion of a particle in the direction transverse to the flow is proportional to the slope of the streambed, as measured by the tangent of the angle with the horizontal, and in the direction of flow is proportional to the depth of the stream. The shape found in this way is a cosine function. In Mr. Swain's work the tangent was replaced by the sine of the angle to give a closer representation of the forces acting on the particle as described in your January 21, 1949, memorandum. The channel found by Mr. Swain is wider than that found in the writer's original memorandum, and its shape is expressed in terms of an elliptic integral. In the work to be described herein the formulation is carried one step further to account for the variation of the normal force component which holds the particle against the bed of the stream. It will be noted, however, that the channel shape obtained in this case is identical with that obtained by Mr. Swain. This appears to be due to the compensating nature of the small differences in the basic formulations. The mathematical work is developed further in this memorandum to provide formulas for computing the cross-sectional area and the wetted perimeter of the stable cross-section.

Assumptions

3. The problem of finding a stable channel shape is here based upon the following assumptions:

a. The particle is held against the bed by the component of the submerged weight of the particle acting normal to the bed.

b. At the edge of the stream the particles are at the angle of repose under the action of gravity.

c. The drag force acting on a particle by the flowing stream is proportional to the depth of water above the particle.

d. The forces tending to cause motion along the streambed are the drag force of the flowing water acting in the direction of flow and the component of the gravity force acting on the submerged particle in a direction transverse to the direction of flow.

e. The ratio of the resultant of the forces acting on the particle along the streambed to the component of the gravity force normal to the bed is equal to the tangent of the angle of repose for the material.

f. The particles are everywhere in a state of incipient instability.

Formulation

4. The cross-section of a channel is shown in Figure 1.

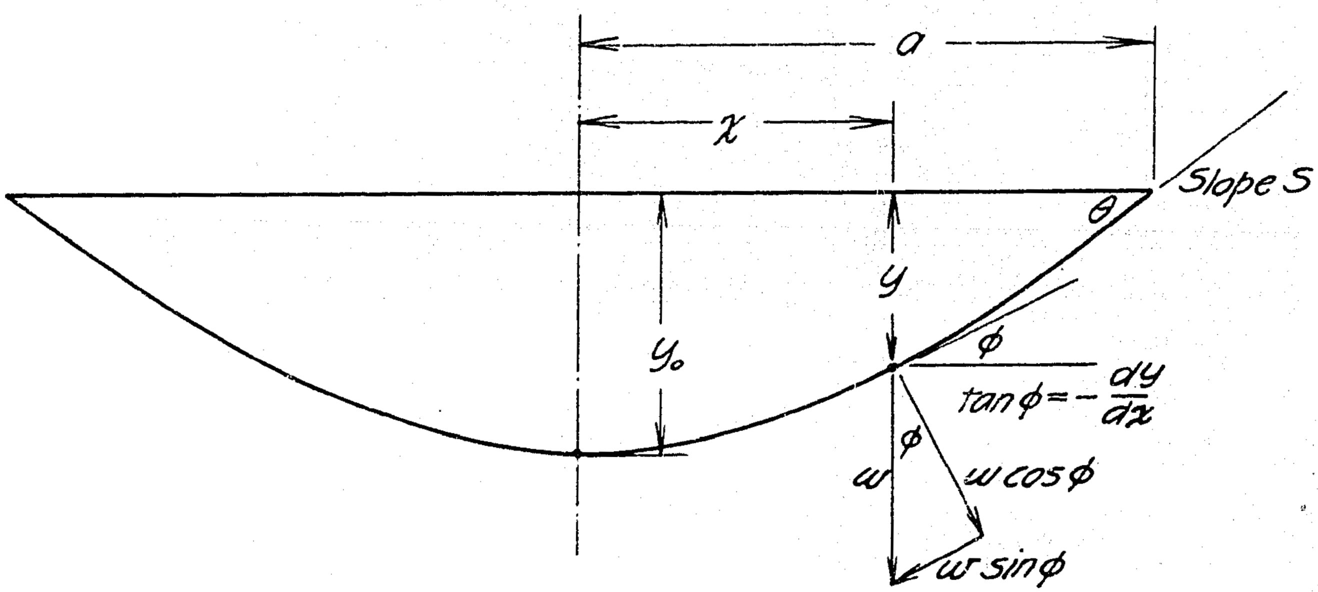

Figure 1

If W represents the submerged weight of a particle, then the force acting along the bed transverse to the channel on a particle at x is $W \sin \phi$. The particle at $x = 0$ is acted on by a drag force $W \sin \theta$ which, by assumption, is required to bring it to the point of incipient instability. At x it is $W \sin \theta (\frac{y}{y_o})$. The resultant of the forces acting along the bed at x is

$$\sqrt{W^2 \sin^2 \phi + W^2 \sin^2 \theta (\frac{y}{y_o})}$$

Then the requirement that the particles everywhere are on the verge of motion is

$$\sqrt{\frac{W^2 \sin^2 \phi + W^2 \sin^2 \theta (\frac{y}{y_o})}{W \cos \phi}} = \tan \theta$$

If $\tan \theta = S$
and also

$$\tan \phi = - \frac{dy}{dx}$$

$$\sin \phi = - \frac{\frac{dy}{dx}}{\sqrt{1 + (\frac{dy}{dx})^2}}$$

$$\cos \phi = \frac{1}{\sqrt{1 + (\frac{dy}{dx})^2}}$$

Then after substitution and rearrangement

$$\sqrt{(\frac{dy}{dx})^2 + S^2 (1 + (\frac{dy}{dx})^2)(\frac{y}{y_o})^2} = S$$

After squaring both sides this becomes

$$(\frac{dy}{dx})^2 + S^2(\frac{y}{y_o})^2 + S^2(\frac{dy}{dx})^2(\frac{y}{y_o})^2 = S^2 \quad \text{.........(1)}$$

This can be put in the form

$$-\frac{\sqrt{1 + S^2(\frac{y}{y_o})^2}}{\sqrt{1 - (\frac{y}{y_o})^2}} \frac{dy}{dx} = S$$

From which, by integration

$$-\int_{y_o}^{y} \frac{\sqrt{1 + S^2(\frac{y}{y_o})^2}}{\sqrt{1 - (\frac{y}{y_o})^2}} dy = Sx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

It is now necessary to evaluate the integral appearing in this equation. As a preliminary step let

$$u = \frac{y}{y_o} \qquad du = \frac{dy}{y_o} \qquad dy = y_o\, du$$

Then the integral becomes

$$-y_o \int_{1}^{u} \frac{\sqrt{1 + S^2u^2}}{\sqrt{1 - u^2}} du$$

To reduce this to one of the standard elliptic integrals, let

$$u = \cos\alpha \qquad du = -\sin\alpha\, d\alpha$$

$$\sqrt{1 - u^2} = \sin\alpha$$

Then by substitution

$$-y_o \int_1^u \frac{\sqrt{1 + S^2u^2}}{\sqrt{1 - u^2}} \, du = y_o \sqrt{1 + S^2} \int_o^{\alpha} \sqrt{1 - \frac{S^2}{1 + S^2} \sin^2\alpha} \, d\alpha$$

This is an elliptic integral of the second kind and the solution of the differential equation can be put in the form

$$\left(\frac{x}{y_o}\right) = \frac{1}{K} E\,(K,\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where

$$K = \frac{S}{\sqrt{1 + S^2}}$$

and

$$E\,(K,\alpha) = \int_o^{\text{arc cos}\,\frac{y}{y_o}} \sqrt{1 - K^2 \sin^2\alpha} \, d\alpha \quad \ldots\ldots\ldots\ldots \quad (4)$$

The function $E\,(K,\alpha)$ has been tabulated.*

To obtain the perimeter, let

$$dp = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

where p represents the length of curve from o to x.

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*See, for example, "A Short Table of Integrals" by Peirce or "Tables of Functions" by Jahnke and Emde.

Then, from (1).

$$dp = \sqrt{1 + \frac{1 + S^2\left(\frac{y}{y_o}\right)^2}{S^2\left(1 - \left(\frac{y}{y_o}\right)^2\right)}}\; dy$$

Or

$$dp = \sqrt{\frac{1 + S^2}{S^2}}\;\sqrt{\frac{1}{1 - \left(\frac{y}{y_o}\right)^2}}\; dy$$

As before, let $u = \frac{y}{y_o}$; then

$$dp = -y_o \sqrt{\frac{1 + S^2}{S^2}}\;\frac{du}{\sqrt{1 - u^2}}$$

By integration

$$p = -y_o \sqrt{\frac{1 + S^2}{S^2}}\;\text{arc sin}\; u \Big]_1^o$$

Or

$$p = \frac{y_o \pi}{2}\sqrt{\frac{1 + S^2}{S^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The area of half the cross-section is:

$$a = \int_o^x y\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Or from

$$a = -y_o \int_1^o \left(\frac{y}{y_o}\right) \sqrt{\frac{1 + S^2\left(\frac{y}{y_o}\right)^2}{S^2\left(1 - \left(\frac{y}{y_o}\right)^2\right)}}\, dy$$

Or with $u = \frac{y}{y_o}$

$$a = -y_o^{\,2} \int_1^o u \sqrt{\frac{1 + S^2u^2}{S^2(1 - u^2)}}\, du$$

To evaluate this integral, let

$$Z^2 = 1 - u^2 \qquad u^2 = 1 - Z^2 \qquad du = -\frac{Z}{u}\, d$$

Which, on substitution, transforms the above integral to

$$a = y_o^{\,2} \int_o^1 \sqrt{\frac{(1 + S^2)}{S^2} - Z^2}\, dZ$$

If $\frac{1 + S^2}{S^2} = m^2$, the required integral is

$$a = y_o^{\,2} \left[\frac{1}{2}\sqrt{m^2 - 1} + \frac{m^2}{2} \text{ arc sin } \left(\frac{1}{m}\right) \right] \quad \ldots\ldots(8)$$

5. Summary of formulas. The stable shape is given by

$$\left(\frac{x}{y_o}\right) = \frac{1}{K} E\,(K, \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where

$$K = \frac{S}{\sqrt{1 + S^2}}$$

and

$$E\,(K, \alpha) = \int_0^{\arc\cos\left(\frac{y}{y_o}\right)} \sqrt{1 - K^2 \sin^2 \alpha}\; d\alpha \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

The perimeter of the whole section is $2p = P$. Then

$$P = y_o \pi\, m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

where

$$m = \frac{\sqrt{1 + S^2}}{S}$$

The area of the whole cross-section is $2a = A$. Then

$$A = y_o^{\,2} \left[\sqrt{m^2 - 1} + m^2 \arc\sin\left(\frac{1}{m}\right) \right] \quad \ldots\ldots\ldots \quad (10)$$

The top width of the stream is given by

$$T = \frac{2}{K}\, E\,\left(K, \frac{\pi}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

The elliptic integral

$$E\,\left(K, \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - K^2 \sin^2 \alpha}\;\; d\alpha \quad \ldots\ldots\ldots \quad (12)$$

is known as the complete elliptic integral of the second kind. Tabular values may be found in the references given previously.

6. A brief tabulation of values is given below with the hydraulic radius $R = \frac{A}{P}$.

Table 1

$\underline{S}$	P/y_o	A/y_o^2	R/y_o	T/y_o
1	4.4429	2.5708	0.5786	3.8200
2/3	5.6636	3.4110	0.6023	5.1992
1/2	7.0248	4.3180	0.6147	6.6595
1/3	9.9346	6.2280	0.6269	9.6782

Robert E. Glover