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* UNITED STATES *
* DEPARTMENT OF THE INTERIOR *
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* Technical Memorandum No. 336 *
* Hydraulic Laboratory Report No. Hyd.-2.2 *
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* THEORY OF THE FLOW NET *
* OF FLUID MOTION IN TWO DIMENSIONS *
* by *
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* ----- *
* Denver, Colorado *
* May 22, 1933 *

UNITED STATES
DEPARTMENT OF THE INTERIOR
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MEMORANDUM TO CHIEF DESIGNING ENGINEER
SUBJECT: THEORY OF THE FLOW OF
FLUID MOTION IN TWO DIMENSIONS

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TECHNICAL MEMORANDUM No. 336

Denver, Colorado

May 22, 1933

(PRICE \$0.25)

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THEORY OF THE FLOW NET OF FLUID MOTION IN TWO DIMENSIONS

The motion of a fluid in two dimensions follows the same basic physical laws as does the flow of electricity and magnetism. This phenomenon allows certain hydraulic flow conditions to be analyzed by electrical methods.

Two geometric properties of the so-called flow net permit certain problems to be solved graphically and lend a valuable tool in the application of the electric analogy apparatus. Rectilinear coordinates are chosen for the construction of the flow net. A system of orthogonal trajectories are drawn to form a grid of curvilinear rectangles or squares. One family of curves is called the lines of equal velocity potential and the trajectories are called lines of equal flow. The two above-mentioned geometric properties are as follows:

1. Lines of equal velocity potential intersect lines of equal flow at right angles.
2. All the curvilinear rectangles have a constant ratio of length to breadth. If curvilinear squares are used, the ratio is 1. These two conditions may be expressed mathematically.

In the following discussion, irrotational motion of an ideal fluid is considered. The velocity potential is designated by the function Φ and is defined as:

$$\Phi = \int_A^P (u dx + v dy) \quad (1)$$

where u and v are the x and y components of velocity and the integration is performed from a point A to a point P in the fluid. It is immaterial what path in the fluid is followed in moving from A to P . Holding first x and then y constant the following relations are evident:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad * \quad (2)$$

* Lamb: Hydrodynamics, page 35, 5th Edition, 1924.

It is difficult to form a physical conception of the function ϕ but its mathematical meaning is distinct.

The flow function is designated by ψ and defined

$$\psi = \int_A^P (lu + mv) ds \quad (3)$$

where l and m are direction-cosines of the normal to an element of the curve ds . The integration is performed from a point A to a point P within the moving fluid. Again the path used need not be specified except that it be within the fluid. Using partial derivatives:

$$u = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \quad ** \quad (4)$$

** Lamb: Hydrodynamics, page 60, 5th Edition, 1924.

The differential equation of continuity is a required condition of the problem. It is expressed as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5)$$

If the values of u and v from equations (3) and (4) are substituted and the operation indicated performed, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (6)$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (7)$$

This is merely another form of the equation of continuity and is sometimes expressed

$$\nabla^2 \phi = 0 \quad (6a)$$

$$\nabla^2 \psi = 0 \quad (7a)$$

* Thompson and Tait: Treatise on Natural Philosophy, Volume 1, Part 1, page 148, 1879.

It is convenient in this part of the proof to use complex numbers. The proof of the perpendicularity of the equal velocity potential lines and equal flow lines begins with the conjugate function

$$x + i y = \phi + i \psi$$

in which $i = \sqrt{-1}$. Introducing a symbol for the functions on

each side of the equation, we have:

$$Z = x + iy, \quad W = \phi + i\psi$$

By the four-step rule used in differential calculus to obtain the derivatives, the following relation is derived. Holding first y constant

$$\frac{dW}{dZ} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \quad (8)$$

Then hold x constant, we have

$$\frac{dW}{dZ} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} \quad (9)$$

Therefore, from equations (8) and (9)

$$\frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i \frac{\partial \phi}{\partial y} \quad (10)$$

Now equating the real parts and the imaginary parts

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad (11)$$

and

$$\frac{\partial \psi}{\partial x} = - \frac{\partial \phi}{\partial y} \quad (12)$$

Equations (11) and (12) are known as the Cauchy-Riemann differential equations.

In order to satisfy the conditions of lines of equal velocity potential and lines of equal flow, we must have

$$\phi = f(x, y) = \text{constant} \quad (13)$$

$$\psi = F(x, y) = \text{constant} \quad (14)$$

Consider first $\phi = \text{constant}$. Then, from equations (1) and (2) we see that

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \quad (15)$$

By transposition and division

$$\left(\frac{dx}{dy}\right)_\phi = - \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}} \quad (16)$$

Now, considering $\psi = \text{constant}$, by the same method, we can prove

$$\left(\frac{dx}{dy}\right)_\psi = - \frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \quad (17)$$

When both ϕ and ψ are constant, and if the lines representing these functions intersect at right angles, their slopes must have the following relation

$$\left(\frac{dx}{dy}\right)_\phi = - \frac{1}{\left(\frac{\partial x}{\partial y}\right)_\psi} \quad (18)$$

Therefore:

$$\left(\frac{\partial x}{\partial y}\right)_\phi \cdot \left(\frac{\partial x}{\partial y}\right)_\psi = -1 \quad (18a)$$

Substitute equations (16) and (17) into equation (18a) and we find

$$\left(-\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}} \right) \cdot \left(-\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \right) = -1 \quad (19)$$

Into the latter equation, substitute equations (11) and (12) in the following manner

$$-\left(-\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \right) \cdot \left(-\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \right) = -1 \quad (20)$$

Equation (20) is an identity and therefore completes the proof of the perpendicularity of the lines of equal velocity potential and the lines of equal flow. In electricity these lines are often referred to as equi-potential lines and lines of equal flux.

The proof of the second property of the flow net or that the ratios of length to breadth of the rectangles are constant involves a vector interpretation of Stokes' theorem. This important theorem of hydrodynamics was introduced by Professor George G. Stokes.* Essentially, it is a formula for changing a line integral into a surface integral. Wilson** gives a concise mathematical derivation of the formula. A comprehensive analytical proof is given by Thomson and Tait *** and a more extended analytical treatment is presented by Maxwell.****

* Smith's Prize Examination Papers for 1854.

** Advanced Calculus--E. B. Wilson, Ginn & Co., page 344, 1911.

*** Natural Philosophy--Thomson and Tait, Cambridge, 1912, Part I, Art. 190 (j), page 143.

**** Electricity and Magnetism---J. Clerk Maxwell, Second Edition, Oxford, 1881, Vol. I, Art. 24, page 25.

The vector interpretation of Stokes' theorem is an invaluable tool in the proof of the proposition.

Using the following vector notation: $\bar{v}_x$, $\bar{v}_y$ and $\bar{v}_z$ equal to the components of the vector velocity $\bar{v}$ in the directions x , y , and z , respectively, Stokes' formula may be written:

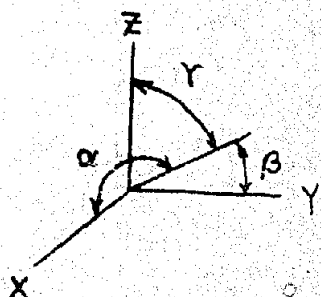


Fig. 1

$$\begin{aligned} \int_0 (\bar{v}_x dx + \bar{v}_y dy + \bar{v}_z dz) = \\ \iint \left[\left(\frac{\partial \bar{v}_z}{\partial y} - \frac{\partial \bar{v}_y}{\partial z} \right) dy dz + \left(\frac{\partial \bar{v}_x}{\partial z} - \frac{\partial \bar{v}_z}{\partial x} \right) dz dx \right. \\ \left. + \left(\frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y} \right) dx dy \right] \end{aligned} \quad (21)$$

Consider now an element of surface ds whose projections on the planes of reference are $dydz$, $dzdx$ and $dx dy$.

Then using the directional cosine notation indicated in the Fig. 1

$$dy dz = dS \cos \alpha \quad (22a)$$

$$dz dx = dS \cos \beta \quad (22b)$$

$$dx dy = dS \cos \gamma \quad (22c)$$

The surface integral of Stokes' formula then reads:

$$\begin{aligned} \iint \left[\left(\frac{\partial \bar{v}_z}{\partial y} - \frac{\partial \bar{v}_y}{\partial z} \right) \cos \alpha + \left(\frac{\partial \bar{v}_x}{\partial z} - \frac{\partial \bar{v}_z}{\partial x} \right) \cos \beta \right. \\ \left. + \left(\frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y} \right) \cos \gamma \right] dS \end{aligned} \quad (23)$$

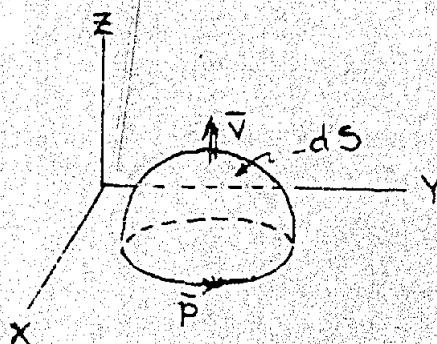


Fig. 2

We now define a new quantity called curl of the vector velocity.

$$\text{Curl } \bar{V} = \lim_{dS \rightarrow 0} \frac{\int_0 \bar{V} d\bar{p}}{\int dS} \quad (24)$$

where $\bar{V}$ is taken normal to the element of surface. It is part of this problem to prove that $\text{curl } \bar{V} = 0$.

In order to show the relation between the curl of the vector velocity and Stokes' formula, an elemental rectangle in the xy plane is analyzed. For the first consideration z may be held constant. The four parts of the line integral of $\bar{V}$ around the rectangle, neglecting higher order derivatives, are:

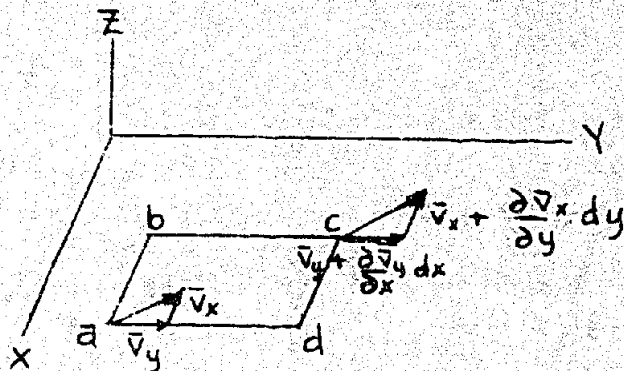


Fig. 3

$$(a \text{ to } b) \quad + \bar{v}_x dx \quad (25a)$$

$$(b \text{ to } c) \quad + (\bar{v}_y + \frac{\partial \bar{v}_y}{\partial x} dx) dy \quad (25b)$$

$$(c \text{ to } d) \quad - (\bar{v}_x + \frac{\partial \bar{v}_x}{\partial y} dy) dx \quad (25c)$$

$$(d \text{ to } a) \quad - \bar{v}_x dy \quad (25d)$$

Adding the parts together we have: *

* Vector Analysis—Richard Gans. Blackie and Son Ltd., 1932, page 42.

$$\begin{aligned} \int \bar{v} d\bar{p} &= \bar{v}_x dx - \bar{v}_y dy - (\bar{v}_x + \frac{\partial \bar{v}_x}{\partial y} dy) dx \\ &+ (\bar{v}_y + \frac{\partial \bar{v}_y}{\partial x} dx) dy = (\frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y}) dx dy \end{aligned} \quad (26)$$

The element of surface $dx dy$ may now be designated dS_z .

Then for the line integral of $\bar{v} d\bar{p}$ in the xy plane, we have:

$$\int_{0xy} \bar{v} d\bar{p} = (\frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y}) dS_z \quad (27)$$

$$\text{Therefore, } (\text{Curl } \bar{v})_z = \lim_{dS_z \rightarrow 0} \frac{\int_0 \bar{v} d\bar{p}}{\int dS_z} = (\frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y}) = \xi = 0 \quad (28a)$$

$$\text{Likewise } (\text{Curl } \bar{v})_y = (\frac{\partial \bar{v}_x}{\partial z} - \frac{\partial \bar{v}_z}{\partial x}) = \eta = 0 \quad (28b)$$

$$(\text{Curl } \bar{v})_x = (\frac{\partial \bar{v}_z}{\partial y} - \frac{\partial \bar{v}_y}{\partial z}) = \xi = 0 \quad (28c)$$

The three above parenthetical expressions are denoted ξ (X_1), η (Eta) and ζ (Zeta) after Helmholtz. They are called the components of vorticity and each must equal zero to satisfy the condition of irrotational motion. As stated by Lamb*

* H. Lamb--Hydrodynamics, 5th Edition, 1925, page 31.

the relative motion of any element consists of pure strain. In other words, the motion in any plane does not induce strain normal to that plane. This is an ideal condition and is one of the fundamental hypotheses of the proof.

The geometrical interpretation of the curl of the vector velocity and remainder of the proof of constant proportionality of the dimensions of the curvilinear rectangles is given by Richardson**

** A Freehand Graphic Way of Determining Stream Lines and Equipotentials--L. T. Richardson, Philosophical Magazine, S. 6, Vol. 15, No. 86, Feb., 1908.

Reference is made to the flow net of infinitesimal rectangles in Fig. 4.

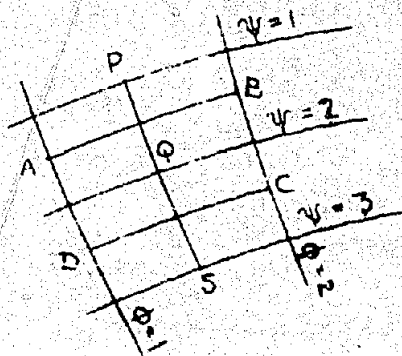


Fig. 4

Lines AB and DC are center lines between $\psi = 1$ and $\psi = 2$ and between $\psi = 2$ and $\psi = 3$, respectively, while PQS is a center line between $\phi = 1$ and $\phi = 2$. All are at right angles. Therefore, PQ and QS are the width of the flow tubes. If the discharge in each tube is the same, then the velocities are inversely as PQ and QS.

$$\bar{V}_{1-2} = \frac{k}{PQ} \quad (29a)$$

$$\bar{V}_{2-3} = \frac{k}{QS} \quad (29b)$$

Taking the line integral around ABCD there is, from the sides AB and CD:

$$\begin{aligned} \int_0 (ABCD) &= AB \times \frac{k}{PQ} - DC \frac{k}{QS} \\ &= k \left(\frac{AB}{PQ} - \frac{DC}{QS} \right) \end{aligned} \quad (30)$$

Richardson designates $\frac{AB}{PQ}$ and $\frac{DC}{QS}$ as the "checker ratios." Thus in words: The line integral of the velocity around ABCD is equal to a constant times the differences in successive checker ratios normal to the flow. As defined above (see equation 24) the curl of the vector velocity is the line integral around a closed curve divided by the area of that curve. The area, as before, is considered approaching zero as a limit. Since the curl V must equal zero, we have:

$$\text{Curl } \vec{V} = K \frac{\left(\frac{AB}{PQ} - \frac{DC}{QS} \right)}{\text{Area } ABCD} = 0 \quad (31)$$

Therefore,

$$\frac{AB}{PQ} - \frac{DC}{QS} = 0 \quad (32)$$

This means that the checker ratios normal to the flow are equal,
which is the condition of proportionality set out to prove.