ENGINEERING MONOGRAPHS No. 29

United States Department of the Interior
BUREAU OF RECLAMATION

CALCULATION OF STRESS FROM STRAIN IN CONCRETE

by Keith Jones

Denver, Colorado
October 1961

65 cents

United States Department of the Interior

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Engineering Monographs

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Technical Information Branch
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Denver, Colorado

ENGINEERING MONOGRAPHS are published in limited editions for the technical staff of the Bureau of Reclamation and interested technical circles in Government and private agencies. Their purpose is to record developments, innovations, and progress in the engineering and scientific techniques and practices that are employed in the planning, design, construction, and operation of Reclamation structures and equipment. Copies may be obtained from the Bureau of Reclamation, Denver Federal Center, Denver, Colorado, and Washington, D. C.

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INTRODUCTION

The purpose of this monograph is to present the mathematical methods used by the Bureau of Reclamation to compute stress from measured strain in concrete. The methods developed are based on studies which indicate that the direct application of Hooke's Law to problems of stress and strain in concrete without modification is in error.

Modification of Hooke's Law is necessitated by the phenomenon of "creep" exhibited by concrete. Concrete, subjected to a constant stress, continues to deform with time. Concurrently, the properties of concrete undergo significant changes with time at least for some months immediately following placement. Consequently, a modification of Hooke's Law must involve time in two aspects, the age of the concrete at the time of its subjection to any particular load, and the time of duration of any particular load. Both of these aspects are developed in the Bureau of Reclamation's engineering laboratories for each concrete tested. The results utilized in the subsequent mathematical analysis are described in this monograph. The Bureau's laboratory investigations have defined the properties of concrete in sufficient detail to permit the derivation of a modified, more accurate, form of Hooke's Law. The use of that modification yields more realistic values of stresses as deduced from measured strains than have been possible heretofore.

PHYSICAL PROPERTIES OF CONCRETE

The relationship of stress to strain in a perfectly elastic material when the stress is less than the elastic limit of that material is expressed by Hooke's Law:

$$\sigma = E\epsilon \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

where σ indicates the magnitude of stress, either positive or negative, E is Young's modulus of elasticity, and ϵ represents unit strain magnitude, either positive or negative, in the direction of the stress. The positive sign denotes tensile stress and increase in length. It has long been recognized that the behavior of concrete is only partly described by Hooke's Law. More specifically, only part of the strain in concrete under applied stress can be related to the applied stress by use of Equation (1). Modifications must be made to Equation (1) to relate strain to stress while involving the change in strain in concrete with time as the applied stress remains unchanged. The use of such a modified formula together with the application of the principle of superposition permits computation of stress varying with time in concrete when sufficient measurements of strain to permit the plotting of a continuous time-strain curve are known.

The comparison between an elastic material and one exhibiting creep properties may be illustrated by an assumed test specimen of an arbitrary material of uniform cross section loaded uniaxially to only a small part of its ultimate strength. If the specimen consists of a perfectly elastic material, the unit deformation $\epsilon = \frac{\sigma}{E}$ takes place instantaneously upon application of the load and remains unchanged with time. Upon removal of the load, the test specimen immediately returns to its original unloaded length regardless of the length of time elapsing between application and removal of the load. The load may be tensile or compressive with the evident deformation and recovery properties differing only in sense.

The test specimen may next be assumed to be composed of a material such as concrete, which exhibits the property of creep. When the load is applied to that specimen, an instantaneous deformation occurs as in the elastic specimen. If the load is immediately removed, the test specimen instantly regains its original unloaded length and the equation $\epsilon = \frac{\sigma}{E'}$ applies. E' is designated as the instantaneous modulus of elasticity of a material exhibiting creep properties. Instead of removing the load, assume that it remains unchanged on the test specimen indefinitely and that measurements of ϵ are continued at predetermined intervals of time. Those measurements will show gradually increasing values of ϵ. Those deformations are equal to the instantaneous deformation upon application of the load plus the creep deformation taking place between the time of application of the load and the time at which ϵ is measured.

Laboratory tests show that the instantaneous modulus of elasticity increases and that the rate of change of creep deformation decreases with increasing age of new concrete. Figure 1 is a set of typical curves obtained from companion concrete specimens each of which was subjected to a load, but with the specimens loaded at progressively increasing ages. Since the curves of Figure 1 are all reduced to unit stress and unit deformation, σ does not appear in their description. It is important to note the use

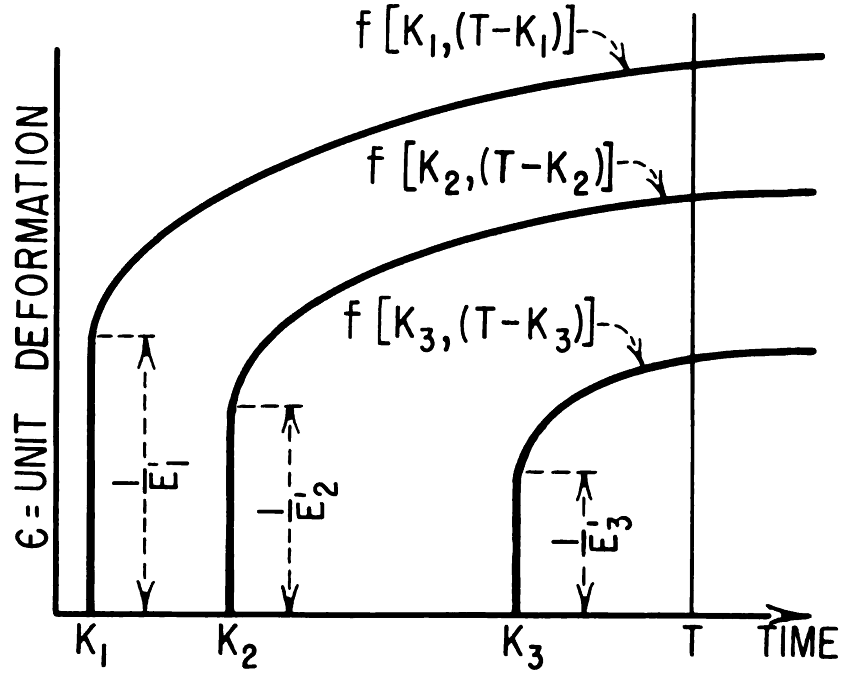

Figure 1 -- Typical creep curves defined by arbitrary creep function.

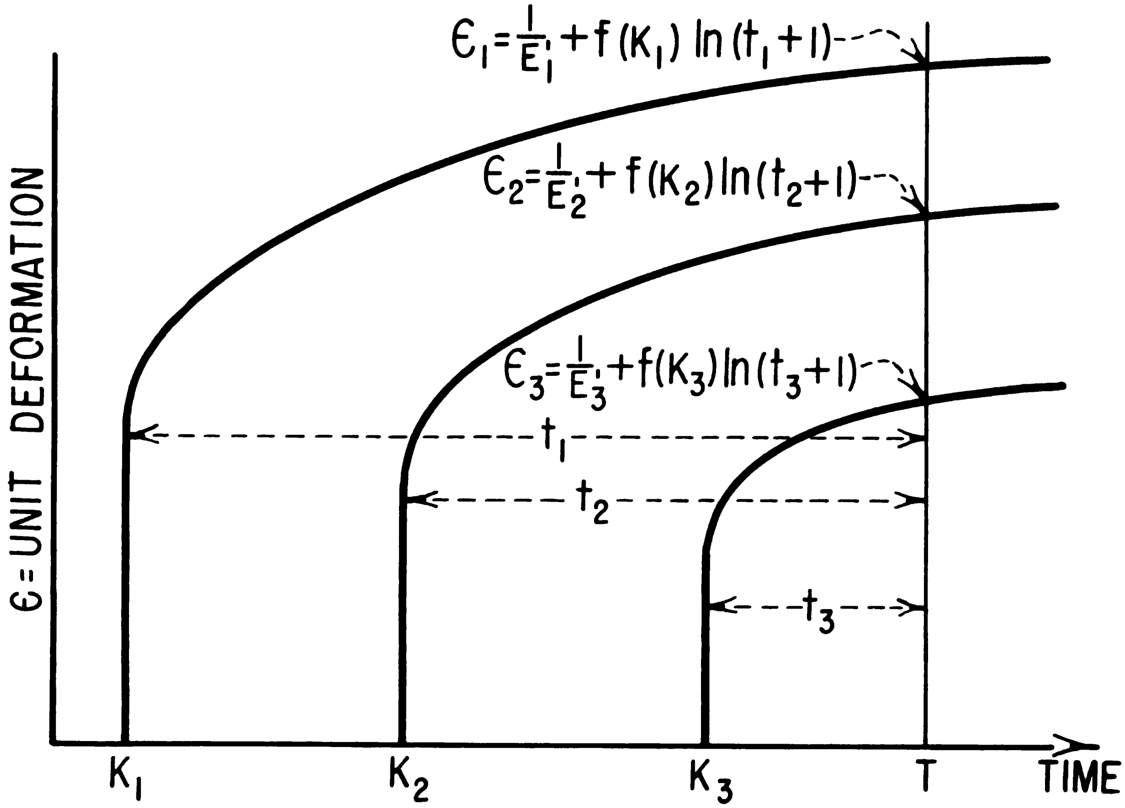

Figure 2 -- Typical creep curves defined by Equation (2).

of two time scales both of which are measured on the same axis, that is, the horizontal axis of Figure 1. Those two time scales indicate first, the age of the concrete at the time of loading, designated K_1, K_2, K_3, on Figure 1; and second, the time elapsing after loading, designated $(T - K_1)$, $(T - K_2)$, and $(T - K_3)$ on the same figure. T designates the age of the concrete at any time at which the deformation due to any or all previously applied loads is desired. For further simplification, t is used to indicate time after loading and is equivalent to the expressions $(T - K_1)$, $(T - K_2)$, $(T - K_3)$ on Figure 1. Figure 1 shows graphically the increase of E' and the gradual decrease in the slope of the curves as the concrete becomes older. It is evident from Figure 1 that K_n (the age of the concrete at a particular loading) plus t (the time after that loading) must equal T (the age of concrete at time t after loading).

It has been determined by curve-fitting procedures that the curve of concrete deformation for any one age at time of loading may be closely represented by an equation of the type:

$$\epsilon = \frac{1}{E'} + f(K) \ln (t + 1) \quad . \; . \; . \; (2)$$

where ϵ, E', and t have the same meanings as already defined, K indicates the age of the concrete at the time of loading, and ln indicates the natural or Naperian logarithm to base e. The quantity (t + 1) replaces t so that the logarithm will have a value at t = 0.

Figure 2 shows the same curves as those of Figure 1, but defined by the more specific Equation (2) instead of arbitrary functions. In practice ϵ is evaluated in millionths of an inch per inch per pound per square inch. K and t are in days. By the generally accepted principle of superposition, the total unit deformation ϵ in a single specimen at time T, subjected to the three loads of Figure 2 is $\epsilon_1 + \epsilon_2 + \epsilon_3$. Curves similar to those of Figure 2 are determined by laboratory tests on specimens of the concrete from the structure to be analyzed. The curves commonly represent the effects of loads placed on the specimens at ages 2, 7, 28, 90, and 365 days. Extrapolation and interpolation procedures are used to obtain curves corresponding to ages before, between, and after those obtained from tests.

Laboratory tests have shown that at low working stresses, when a load on a concrete specimen is reduced or removed at some particular time after application, the amount of instantaneous recovery and the amount and rate of creep recovery are of the same magnitudes as the instantaneous and creep deformations that would be produced by an equal increase or application of load at the same time. Referring to Figure 2, if, at age K_3, for example, the load shown applied at that time had been removed or, equivalently, had been applied in the opposite sense, the resulting deformation would be opposite to that shown and the total deformation at time T would be $\epsilon_1 + \epsilon_2 - \epsilon_3$. That result is, of course, an extension of the accepted principle of superposition.

If the unit deformation due to unit stress is

$$\epsilon = \frac{1}{E'} + f(K) \ln (t + 1) \quad . \; . \; . \; . \; . \; (2)$$

and if the principle of superposition is accepted, then the change in deformation, $\Delta\epsilon$, due to a change in stress, $\Delta\sigma$, must be

$$\Delta\epsilon = \Delta\sigma \left[\frac{1}{E'} + f(K) \ln (t + 1)\right]$$

or

$$\Delta\sigma = \frac{\Delta\epsilon}{\frac{1}{E'} + f(K) \ln (t + 1)} \quad . \; . \; . \; . \; (3)$$

which is a modification of Equation (1) that relates strain changes to stress changes in a material exhibiting creep properties.

APPLICATION OF LABORATORY DATA TO MEASURED STRAINS

By applying Equation (3) to selected values of $\Delta\epsilon$ obtained at proper intervals of time from a curve of measured strains and having determined values of $\frac{1}{E'}$ and f(K) at those intervals from laboratory tests it is possible to compute $\Delta\sigma$ for each selected $\Delta\epsilon$ so that a stress curve corresponding to the measured strain curve can be plotted from the summation of the stress changes. Equation (3) yields stress changes dependent upon strain changes measured at, and occurring between, selected times.

The selected intervals of time during which individual strain changes occur should be short during the early age of the concrete while the elastic and creep properties are changing rapidly and, to reduce the tedious labor required, gradually lengthened as the concrete age increases. The ages of the concrete at times of loading are taken at the beginning of each time interval and the values of $\Delta\epsilon$ selected at the middle of each of those intervals. A load applied at the beginning of any interval is the computed stress increment, $\Delta\sigma$, necessary to produce the measured strain increment, $\Delta\epsilon$, at the middle of that interval. Owing to the creep characteristics of concrete, stress, as a function of strain, at any time is dependent upon the entire past history of stress variation from the time of placement of the concrete. The computation of stress is a step-by-step process. The stress at any time is the summation of all stress increments, $\Delta\sigma$, positive and negative to that time. Those increments are computed by Equation (3) in which the quantity $\frac{1}{E'} + f(K) \ln (t + 1)$ is determined for the age of the concrete at each mid-interval when each $\Delta\epsilon$ is computed.

If it is assumed that a constant modulus of elasticity exists throughout a short time interval and that modulus is used to compute $\Delta\epsilon$ at the middle of the interval, an approximate value will result. If the time intervals used are sufficiently short so that there is little difference between the modulii at the beginning and at the end of the interval the approximation can be made to any required degree of accuracy, and, in the limit, an exact solution would result. That modulus of elasticity, assumed to exist throughout a time interval, is designated E_s, the sustained modulus of elasticity, and is defined as

$$E_s = \frac{1}{\frac{1}{E'} + f(K) \ln (t + 1)} \quad . \; . \; . \; . \; . \; (4)$$

so that $\Delta\sigma = \Delta\epsilon E_s$ is analogous to Equation (1) expressing Hooke's Law for elastic materials.

Each computed $\Delta\sigma$ can be considered to be a load which causes a strain, increasing with time, according to the appropriate curve from the family represented by Figure 2. Multiplication of $\Delta\sigma$ by the ordinates of that curve at the succeeding mid-intervals gives values of $\Delta\epsilon$ due to $\Delta\sigma$. Addition of all values of $\Delta\epsilon$ to any time T gives the total strain resulting from previously applied loads. The difference between that sum and the measured strain at that time is another $\Delta\epsilon$ from which another $\Delta\sigma$ can be computed by Equation (3) and the process repeated and continued to the end of the strain record. Forms have been devised to facilitate the calculation of stress, tabulation of data in order of use, and systematic storage of the calculated values of $\Delta\sigma$ and $\Delta\epsilon$.

Figure 3 shows the actual instantaneous strain plus creep strain curves for a particular concrete as determined in the Bureau of Reclamation concrete laboratory. That figure is a specific example of the general case shown by Figure 2.

For any particular age of concrete at time of loading, the functions $\frac{1}{E'}$ and f(K) reduce to constants. The values of those functions for the five experimental loadings are given in the equations on Figure 3. They were determined by a method of least squares for a fit of the strain function to points sampled from the total number of laboratory measurements made in the tests. Mathematical expressions for $\frac{1}{E'}$ and f(K) are unnecessary, but graphical representation of those functions is given by the two lower curves of Figure 4. To eliminate the need for logarithm tables in interpolation, a third curve representing the elastic and creep strain at t = 999 days after loading (t + 1 = 1000) has been computed from the five experimental points on each of the other two curves and the third curve also plotted on Figure 4.

Using the $\frac{1}{E'}$ and ϵ_{999} curves from Figure 4, instantaneous strain plus creep strain curves for any age of concrete at time of loading, that is, any K can be drawn. Figure 5 shows those curves for a number of ages at times of loading.

The curves of Figures 3 and 5 are members of the same family and the curves for any corresponding age of concrete at time of loading, for example K = 2 days, yield identical information from either figure. The logarithmic form of the equations of the strain curves produces straight lines when plotted on semi-logarithmic paper which is used for purposes of facilitation and convenience only. It would probably be possible, but difficult and inconvenient, to draw all of the required curves on a uniform scale such as Figure 3. Figure 5 was drawn by plotting the ordinates of the $\frac{1}{E'}$ and ϵ_{999} curves from Figure 4 at each desired K. Curves for later ages of loading and longer times after loading, that is, greater values of K and t, can be obtained by extending the curves of Figure 4 by extrapolation or laboratory experiment, and from those extensions the curves of Figure 5 can be extended. Values for $\frac{1}{E_s}$ for any selected K at any t in millionths of an inch per inch per pound per square inch can be taken directly from Figure 5 and the necessary extensions beyond the time limits shown on that figure.

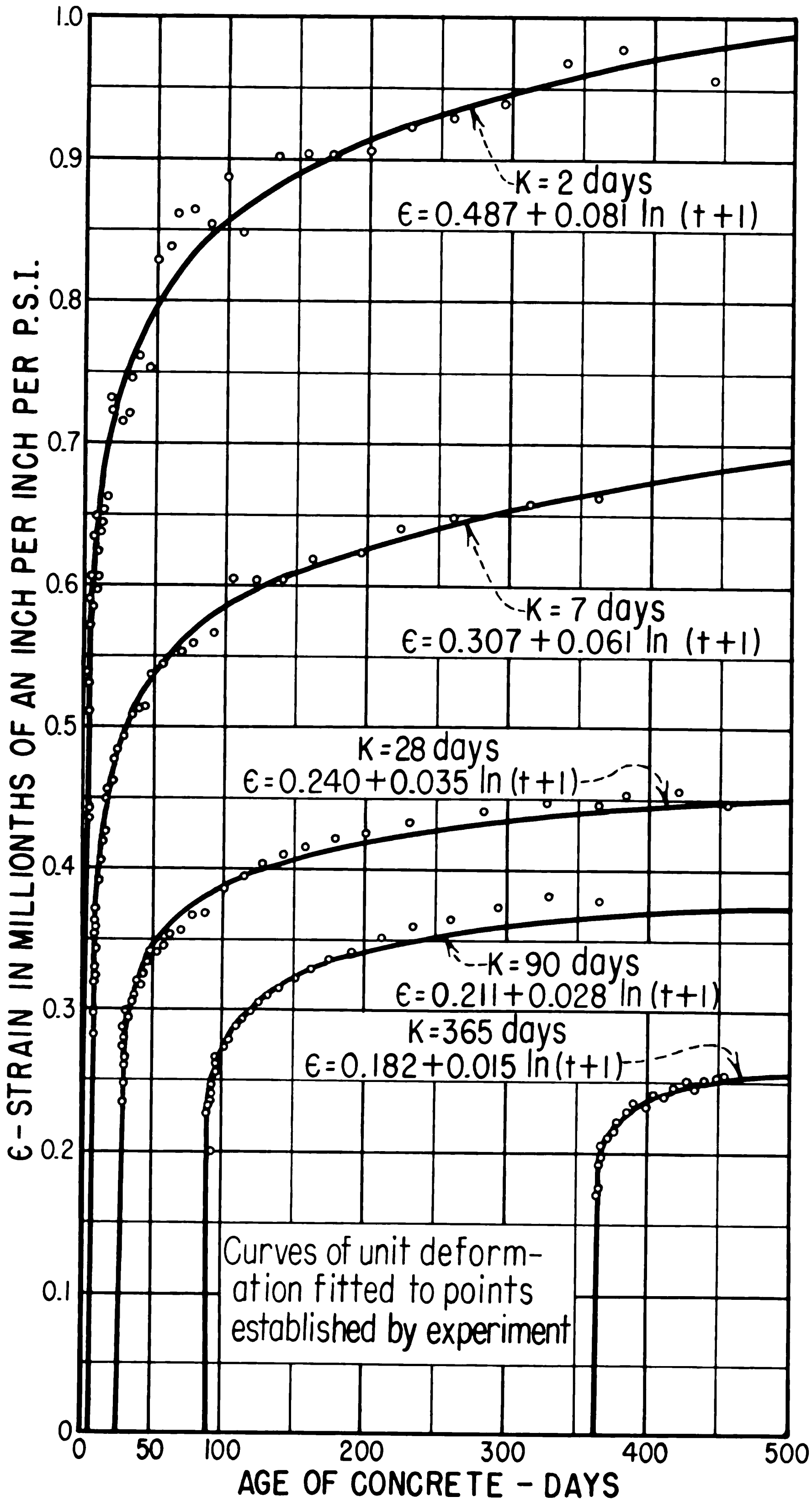

Figure 3 -- Experimental deformation-time curves of a concrete, showing equations derived from the fitted curves.

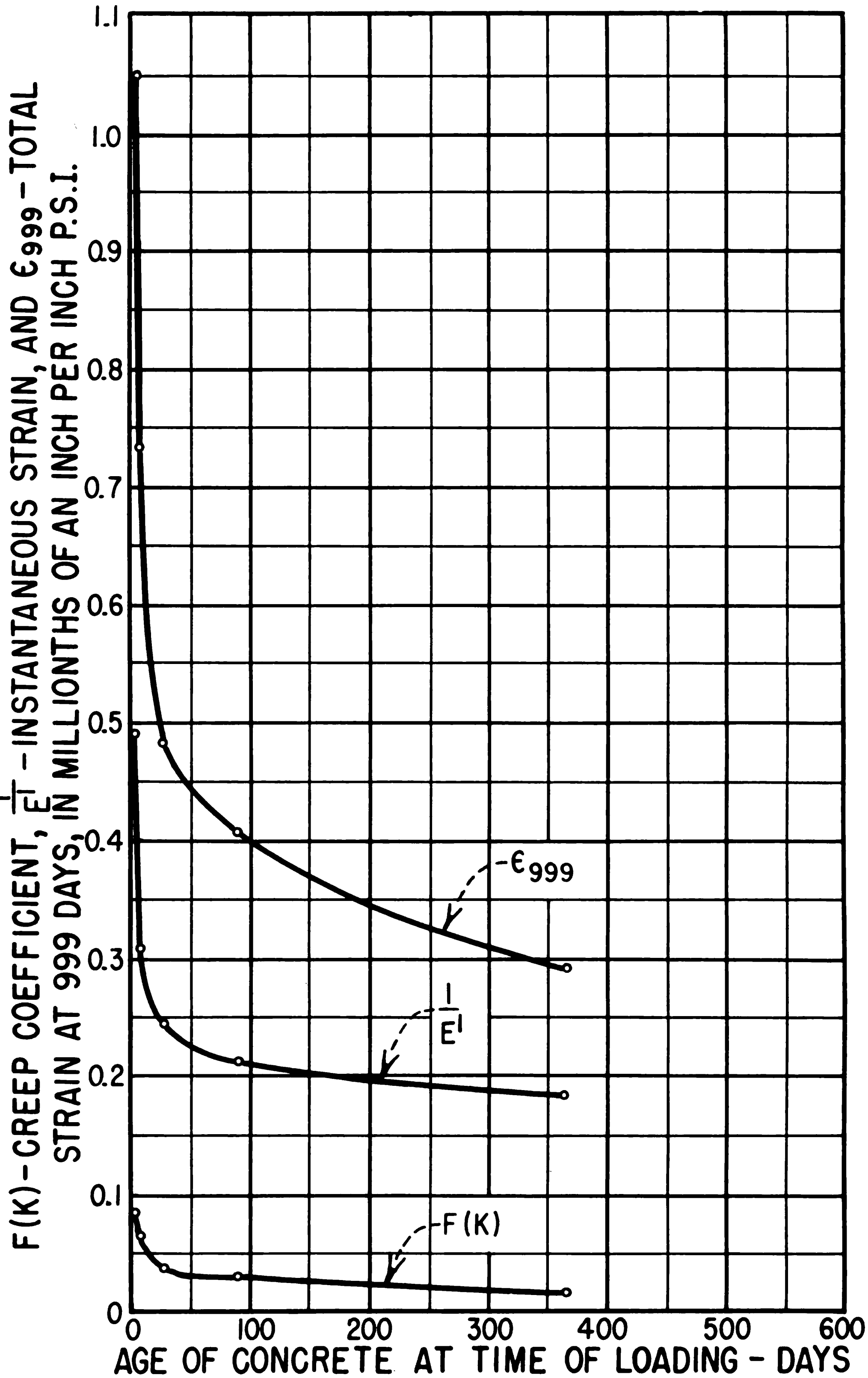

Figure 4 -- Curves of the mathematical expressions used in the equations of Figure 3.

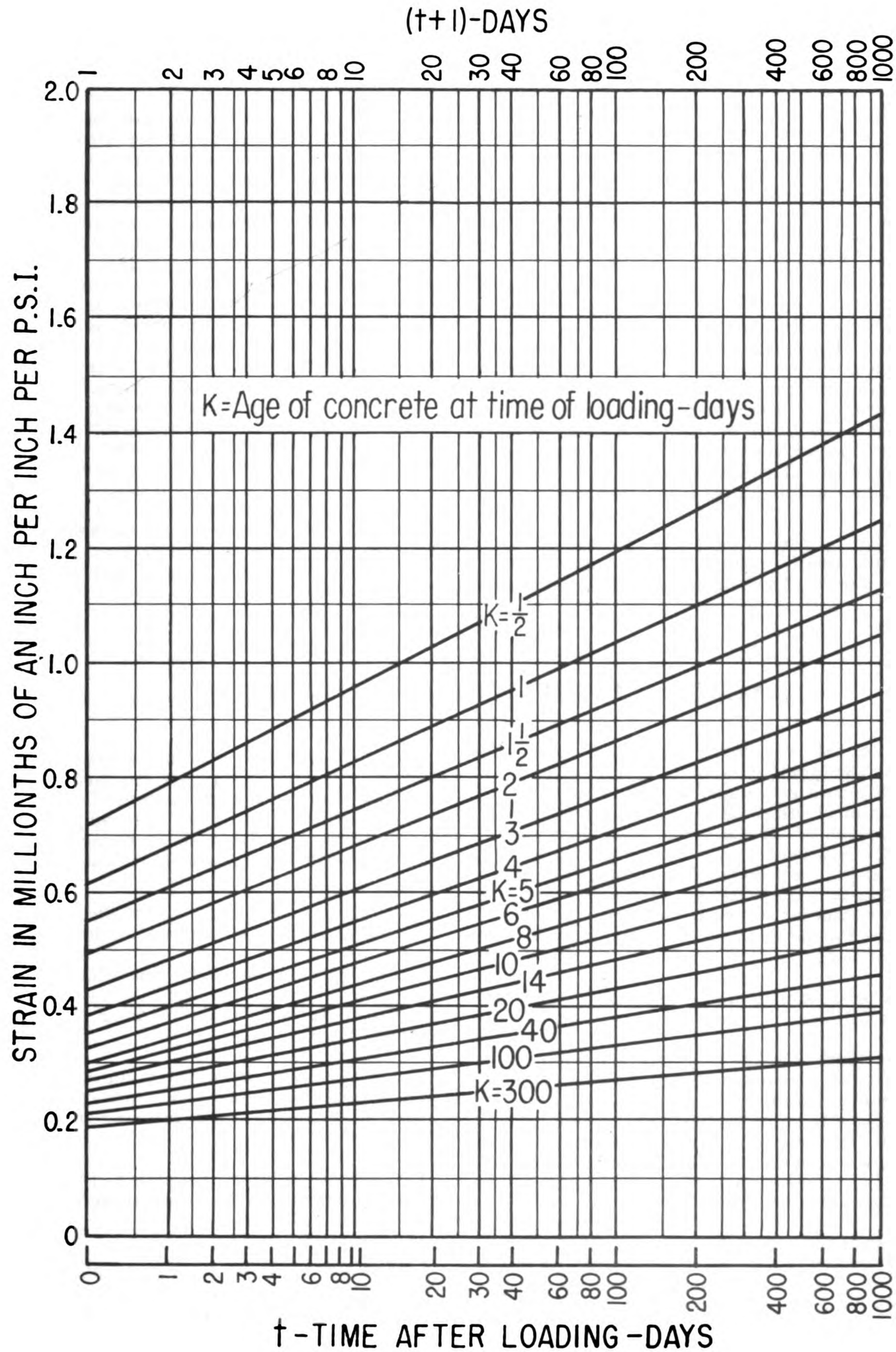

Figure 5 -- Logarithmic representation of the family of the curves of Figure 3, including extrapolated and interpolated members of that family.

Figure 6 is a curve of strain as measured by a strain meter embedded in a concrete dam. Specimens of concrete from the same dam were used in the laboratory to obtain the curves of Figure 3, which, therefore, describe the elastic and creep properties of the same concrete as that in which the strain curve of Figure 6 was measured.

Having a curve of measured strain in concrete, as determined in Figure 6, and the elastic and creep characteristics of the same concrete as described by Figure 5, it is possible, as previously outlined in this monograph, to compute the stress causing the measured strain by the use of a step-by-step process of computing stress increments; summation of all preceding stress increments yields the stress existing at any time.

Table I presents the numerical values of the ordinates of the curves of the families of Figure 3 or 5. The selected ages of concrete at times of loading and at the mid-intervals as used by the Bureau of Reclamation are given in that table for the first 19-1/2 days only. Selection of those ages and, therefore, the lengths of the intervals is somewhat arbitrary. The intervals should be short enough so that all significant changes in the strain curve will be recognized. During the early age of a concrete structure, temperature and construction effects together with the rapidly changing properties of the concrete itself dictate the selection of relatively short intervals. In dams the imposition and relief of loads after construction is completed is a slow process. Consequently, the Bureau uses in its analyses intervals of a maximum length of 90 days after a concrete age of several years. The intervals are gradually lengthened from those shown on Table I to the maximum length. In structures subjected to more rapid changes of load, shorter maximum intervals would be necessary to take cognizance of more frequent strain changes.

ACTUAL COMPUTATION OF STRESS

Table II shows a form designed to facilitate the systematic recording and storage of strain and stress increments and the tabulation of other data necessary to the computation of stress. The form may be extended downward and to the right as far as required to accommodate a strain record of any length. The table also shows a sample stress computation from the strain curve of Figure 6 using the laboratory data summarized in Table I.

The procedure in using the form of Table II is as follows:

Enter under "K-age of concrete at time of loading-days" those ages from Table I. The figures representing those ages should be placed directly above the ends of the vertical lines so that the spaces between the vertical lines may be considered as representing intervals of time elapsing between application of successive loads.

Enter in the first column designated "Age at mid-interval" the numbers in the first column of Table I. Note that those ages are halfway between successive ages of concrete at time of loading.

Enter in the second column designated "Strain at mid-interval ϵ " values from the strain curve at ages corresponding to the first column.

Enter in the fifth column designated "Sustained modulus E_s" the reciprocals of the first number in each column of Table I. The remainder of the stress computation, as shown on Table II, is completed by computation from the data so entered.

Enter zero on the first line of the third column designated "Strain from prior loads."

The value on any line of the fourth column designated "Strain change $\Delta\epsilon$ " is the value on the same line in the second column minus that in the third.

The value on any line of the sixth column designated "Stress change $\Delta\sigma$" is the product of the values on the same line in the fourth and fifth columns.

The seventh column designated "Stress at mid-interval σ " is the summation of the sixth column.

The numbers in the triangular part of Table II are increments of strain due to the creep effects of the stress increments. Those strain increments on any one line are obtained by multiplying the value in the same line of the sixth column consecutively by the numbers in the appropriate column of Table I and recording the results to the nearest whole number.

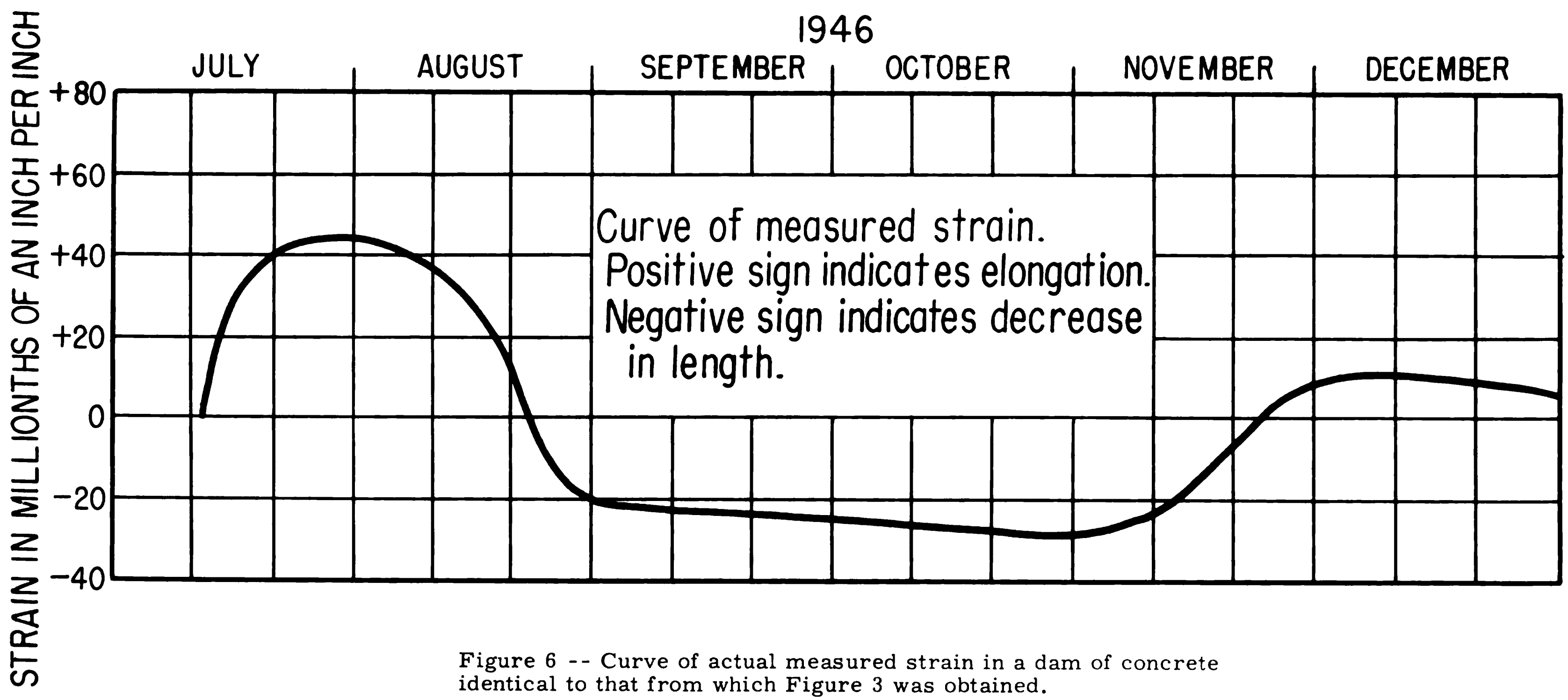

Figure 6 -- Curve of actual measured strain in a dam of concrete identical to that from which Figure 3 was obtained.

Unit strain, ϵ , in millionths of an inch per inch per pound per square inch, produced at successive mid-intervals by loads applied at the beginning of each interval.

This Table was obtained from Fig. 5.

Example: To find ϵ in the concrete at age 9 days due to a load applied at, and maintained since, age 2 days: T=9, K=2. To enter Fig. 5, t is needed.

t=T-K=9-2=7 days.

At the intersection of t=7 and K=2 on Fig. 5 read 0.656 and enter that value in the K=2.00 column and age=9.00 line on this table.

K--Age of concrete at time of loading--days

Age at mid-interval--days	0.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00	7.00	8.00	10.00	12.00	14.00	16.00	18.00
0.875	0.662																
1.25	0.689	0.630															
1.75	0.716	0.661	0.562														
2.25	0.738	0.684	0.591	0.507													
2.75	0.756	0.703	0.612	0.533	0.466												
3.25	0.771	0.718	0.629	0.554	0.492	0.439											
3.75	0.784	0.732	0.643	0.570	0.512	0.464	0.414										
4.50	0.800	0.748	0.661	0.590	0.535	0.491	0.448	0.409									
5.50	0.819	0.767	0.680	0.610	0.557	0.517	0.477	0.443	0.375								
6.50	0.834	0.782	0.695	0.626	0.574	0.535	0.497	0.467	0.410	0.350							
7.50	0.848	0.796	0.708	0.640	0.589	0.550	0.513	0.485	0.432	0.383	0.334						
9.00	0.865	0.813	0.725	0.656	0.606	0.568	0.532	0.505	0.456	0.413	0.376	0.340					
11.00	0.884	0.831	0.743	0.673	0.624	0.588	0.551	0.525	0.478	0.439	0.407	0.380	0.324				
13.00	0.900	0.847	0.757	0.688	0.639	0.602	0.567	0.541	0.496	0.457	0.428	0.403	0.360	0.313			
15.00	0.913	0.860	0.770	0.700	0.651	0.615	0.579	0.553	0.509	0.472	0.443	0.420	0.381	0.346	0.300		
17.00	0.925	0.872	0.781	0.711	0.662	0.625	0.590	0.564	0.520	0.483	0.455	0.433	0.396	0.365	0.332	0.293	
19.50	0.938	0.885	0.793	0.723	0.674	0.637	0.602	0.575	0.532	0.495	0.467	0.445	0.410	0.382	0.354	0.327	0.295

Table I -- Laboratory strain data from Figure 5 at selected mid-intervals.

Age at mid-interval	Strain at mid-interval ϵ	Strain from prior loads	Strain change $\Delta\epsilon$	Sustained modulus E_s	Stress change $\Delta\sigma$	Stress at mid-interval σ	K--Age of concrete at time of loading--days: 1–1½	1½–2	2–2½	2½–3	3–3½	3½–4	4–5	5–6	6–7	7–8	8–10	10–12	12–14	14–16	16–18
0.875	+ 3	0	+ 3	1.511	+4.533	+5	+3	+3	+3	+3	+3	+4	+4	+4	+4	+4	+4	+4	+4	+4	+4
1.25	+ 6	+ 3	+ 3	1.587	+4.761	+10		+3	+3	+3	+3	+3	+4	+4	+4	+4	+4	+4	+4	+4	+4
1.75	+13	+ 6	+ 7	1.779	+12.453	+22			+7	+8	+8	+8	+8	+8	+9	+9	+9	+9	+9	+10	+10
2.25	+20	+13	+ 7	1.972	+13.804	+36				+7	+8	+8	+8	+8	+9	+9	+9	+9	+9	+10	+10
2.75	+22	+21	+ 1	2.146	+2.146	+38					+1	+1	+1	+1	+1	+1	+1	+1	+1	+ 1	+ 1
3.25	+26	+23	+ 3	2.278	+6.834	+45						+3	+3	+4	+4	+4	+4	+4	+4	+ 4	+ 4
3.75	+30	+27	+ 3	2.415	+7.245	+52							+3	+3	+4	+4	+4	+4	+4	+ 4	+ 4
4.50	+33	+31	+ 2	2.457	+4.914	+57								+2	+2	+2	+2	+3	+3	+ 3	+ 3
5.50	+34	+34	0	2.667	0	+57									0	0	0	0	0	0	0
6.50	+36	+37	- 1	2.857	-2.857	+54										-1	-1	-1	-1	- 1	- 1
7.50	+38	+36	+ 2	2.994	+5.988	+60											+2	+2	+3	+ 3	+ 3
9.00	+40	+38	+ 2	2.941	+5.882	+66												+2	+2	+ 2	+ 3
11.00	+42	+41	+ 1	3.086	+3.086	+69													+1	+ 1	+ 1
13.00	+43	+43	0	3.195	0	+69														0	0
15.00	+44	+45	- 1	3.333	-3.333	+66															- 1

Increments of strain

strain, ϵ and $\Delta\epsilon$, is in millionths of an inch per inch

stress, σ and $\Delta\sigma$, is in pounds per square inch

E_s is in pounds per square inch multiplied by 10^{-6}

Table II -- Calculation of stress by strain increment method from Figure 6 and Table I.

The number to be entered on any line of the third column "Strain from prior loads" is obtained as the sum of all strain increments above that line in the corresponding column of the triangular part of the table. For example, on Table II, the strain from prior loads at the 2-3/4-day age at mid-interval is the sum of the column between lines K = 2-1/2 and K = 3 and is: 3 + 3 + 8 + 7 = 21. That result is entered on the fifth line of the third column.

Some practice is necessary to familiarize a computer with this strain increment method of computing stress. After facility is attained in its use, the method is entirely practical if the total amount of work to be done is not too great. The stress, as given by the seventh column of Table II, is shown on Figure 7 and is the curve designated "Strain increment method." That curve extends somewhat beyond the 15-day limit given in the sample calculation of Table II.

When long strain records from large numbers of strain meters are to be analyzed the strain increment method of stress calculation becomes tedious. Experience has shown that the method is not well adapted for use by punched-card calculators. Attempts to use punched-card calculators to compute stresses in concrete resulted in a modified method which has been used successfully on those machines and on desk calculators as well. That modified method is designated, for reasons which will become apparent in its description, as the "Average Logarithm Method."

THE AVERAGE LOGARITHM METHOD

The principal objection to the use of the strain increment method by punched-card calculating machines lies in the computation, accumulation, and storage of the progressive strain increments shown in the triangular tabulation on the stress computation sheets such as Table II. To overcome this objection the average logarithm method has been derived and depends upon the development of a function which will accumulate and store all of the strain increments before and including any one interval in one operation. Equation (3) is the fundamental equation used in both methods of computing stress from measured strain in concrete and is:

$$\Delta\epsilon = \Delta\sigma \left[\frac{1}{E'} + f(K) \ln (t + 1)\right] \cdot \cdot \cdot \quad (3)$$

Stress, at any time, is the summation of all $\Delta\sigma$ preceding that time. . Total strain at any time is likewise the summation of all $\Delta\epsilon$ to that time. Equation (3) may be expanded to its complete form, that is, the form showing the summation of strain increments for all intervals, i, from i = 1 to i = n, and there results:

$$\epsilon = \sum_{i=1}^{n} \Delta\epsilon_i$$

$$= \sum_{i=1}^{n} \left[\frac{1}{E'_i} + f_i(K) \ln (t + 1)\right] \Delta\sigma_i . (5)$$

Equation (5) represents the sum of all strain increments in any column of the triangular part of Table II. That sum is the number entered on the nth line of the third column of Table II.

Referring to Equation (5) and to Table II it may be deduced that, if the logarithmic term of the equation were replaced by some new term, that new term may possibly be selected so as to be a factor representing all the ln (t + 1) for any column in the triangular part of Table II, that is, for all intervals previous to any T. The concept of using an average logarithm was arrived at intuitively. Several variations of the concept were used and the resulting computed stresses were compared with those found by the strain increment method for each of a number of different strain records. Some of those records were from strain meters in actual service in an existing dam, and others were entirely artificial and were made to provide tests for agreement of the methods in possible, although hypothetical, situations. It was determined by those trials that the results, when the term ln (t + 1) was replaced by ln avg (t + 1), were in better agreement with those of the strain increment method than were other variations of the logarithmic term which were tried.

Accordingly, Equation (3) was rewritten as

$$\Delta\sigma_n = E_s \left[\epsilon_n - \sum_{i=1}^{n} \frac{1}{E_i} \Delta\sigma_{i-1} - \ln \text{avg } (t + 1)_i \sum_{i=1}^{n} f(K_i) \Delta\sigma_{i-1}\right] \quad (6)$$

Table III shows a form devised for use in applying Equation (6). The calculation of stress demonstrated on that table is for the identical strain record and concrete properties used in the computation on Table II. The resulting stress curve is plotted on Figure 7 to show a comparison with the stresses computed in Table II.

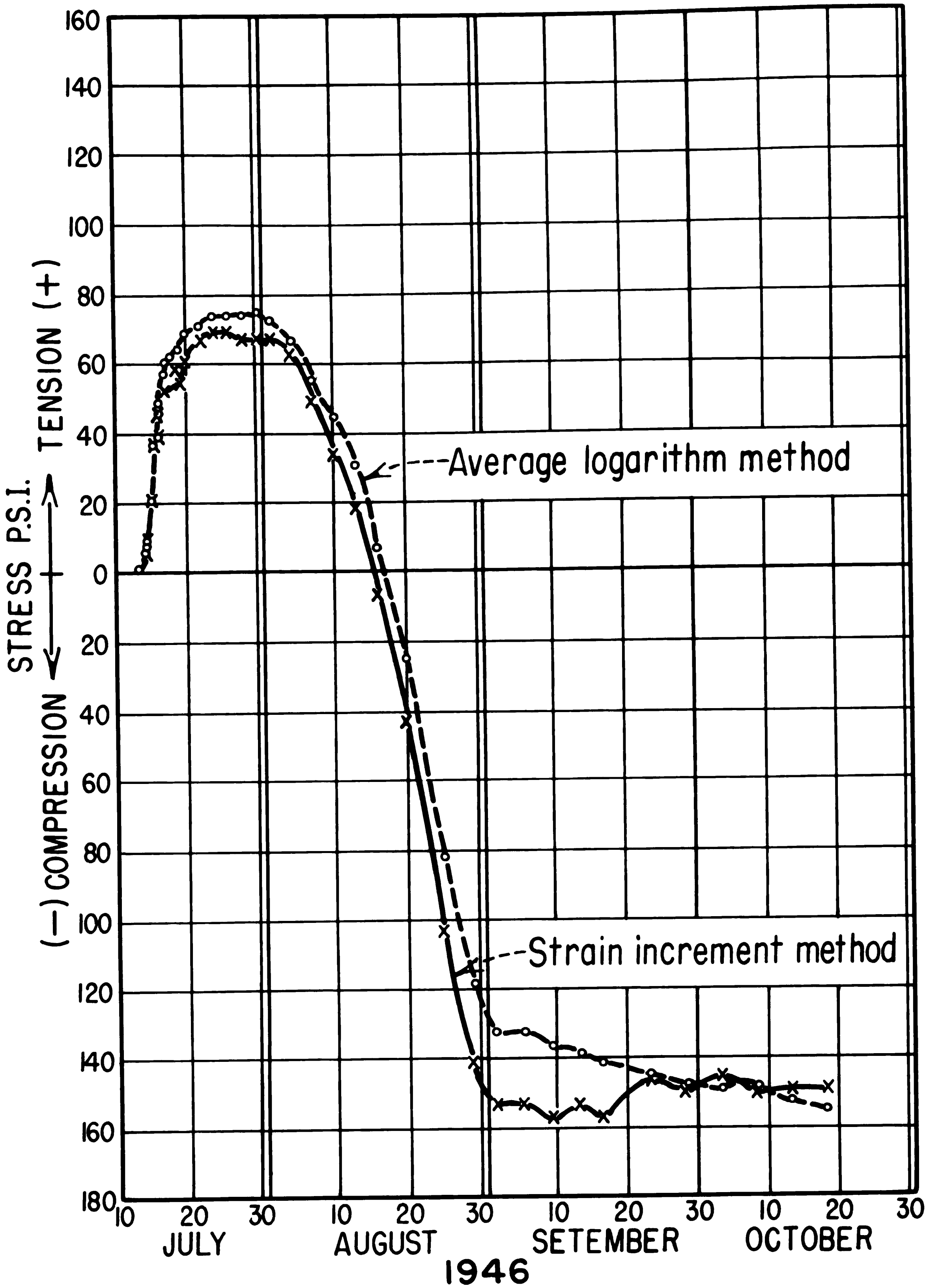

Figure 7 -- Comparison of the stresses calculated by two methods from the strain curve of Figure 6.

(1) Age at mid-int.	(2) f(K)	(3) $\frac{1}{E'}$	(4) ln avg(t+1)	(5) E_s	(6) ϵ	(7) $\Sigma f(K)\Delta\sigma$	(8) $\Sigma \frac{1}{E'} \Delta\sigma$	(9) $(5)\left[\frac{(6)-(8)-(7)(4)}{\Delta\sigma}\right]$	(10) σ	(11) Date
0.875	---	---	---	1.511	+ 3	0	0	+ 4.53300	+ 4.53	7-15-46
1.25	0.090	0.630	0.12883	1.587	+ 6	+0.40797	+ 2.85579	+ 4.90645	+ 9.44	7-15-46
1.75	0.085	0.544	0.51103	1.779	+13	+0.82502	+ 5.52490	+ 12.54816	+21.99	7-16-46
2.25	0.081	0.490	0.66140	1.972	+20	+1.84142	+11.67350	+ 14.01812	+36.01	7-16-46
2.75	0.078	0.450	0.78846	2.146	+22	+2.93483	+17.98165	+ 3.65754	+39.66	7-17-46
3.25	0.075	0.423	0.89947	2.278	+26	+3.20915	+19.52879	+ 8.16590	+47.83	7-17-46
3.75	0.073	0.398	0.99853	2.415	+30	+3.80526	+22.77882	+ 8.26295	+56.09	7-18-46
4.50	0.070	0.380	1.16901	2.457	+33	+4.38367	+25.91874	+ 4.80764	+60.90	7-18-46
5.50	0.067	0.350	1.36525	2.667	+34	+4.70578	+27.60141	- 0.06929	+60.83	7-19-46
6.50	0.064	0.325	1.52061	2.857	+36	+4.70135	+27.57889	+ 3.63465	+64.47	7-20-46
7.50	0.061	0.310	1.64952	2.894	+38	+4.92306	+28.70563	+ 3.51400	+67.98	7-21-46
9.00	0.058	0.300	1.84253	2.941	+40	+5.12687	+29.75983	+ 2.33445	+70.32	7-23-46
11.00	0.053	0.290	2.05757	3.086	+42	+5.25060	+30.43682	+ 2.34443	+72.66	7-25-46
13.00	0.050	0.280	2.22656	3.195	+43	+5.36782	+31.09326	- 0.14387	+72.52	7-27-46
15.00	0.046	0.270	2.36556	3.333	+44	+5.36120	+31.05442	+ 0.87771	+73.40	7-29-46
17.00	0.043	0.263	2.48357	3.413	+45	+5.39894	+31.28526	+ 1.04469	+74.44	7-31-46
19.50	0.041	0.258	2.62315	3.390	+45	+5.44177	+31.55479	- 2.81156	+71.63	8- 2-46

Table III -- Calculation of stress by average logarithm method from Figures 4 and 6.

DETAILS OF THE USE OF THE AVERAGE LOGARITHM METHOD

A form, such as Table III, is necessary to keep the data and calculations in proper order if it is desired to use that method with desk calculators.

The first column is a tabulation of the selected ages at the mid-intervals of loading.

The second column is obtained from the f(K) curve of Figure 4 at ages corresponding to those listed in the first column.

The third column is obtained from the $\frac{1}{E'}$ curve of Figure 4 at the same ages as the values in the first and second columns.

The fourth column is the ln of the quantity obtained by calculation from the following formula:

$$\text{avg } (t + 1) = \frac{1}{n}\left[nT_n - \sum_{i=1}^{n} K_i\right] + 1$$

where n = the number of the interval at which ln avg (t + 1) is required.

T_n = age at the mid-interval of the same interval n

K_i = age at loading = age at beginning of interval n.

For example, to compute avg (t + 1) at n = 4 on Table III:

$$K_n = 2$$

$$T_n = 2.25$$

$$nT_n = 4 \times 2.25 = 9.00$$

$$\sum_{i=1}^{n} K_i = 0.75 + 1 + 1.5 + 2 = 5.25$$

$$\text{avg } (t + 1) = \frac{9 - 5.25}{4} + 1 = 1.9375$$

$$\ln 1.9375 = \log_e 1.9375 = 0.66140$$

The value 0.66140 is entered on the fourth line of Column 4.

E_s, Column 5, is essentially the same as that used in the strain increment method. In that method it is obtained as the reciprocal of the first number of each column of Table I; Table I is taken from Figure 5. Since in this average logarithm method the necessity for Figure 5 and Table I to provide unit strain increments has been avoided, it is necessary to compute values of E_s from the equation:

$$E_s = \frac{1}{\frac{1}{E'} + f(K)\ \ln(t + 1)} = \frac{1}{\epsilon} \quad . \ . \ . \quad (4)$$

where $(t + 1) = (T_n - K_n + 1)$ at any interval, and n, T_n, K_n have the same meanings as used in the calculation of Column (4).

Computed values of ϵ may differ very slightly from those obtained graphically from Figure 5. The values of E_s used on Table III were copied directly from those on the stress computation of Table II and were, therefore, obtained from Figure 5 and Table I.

Since the first five columns of Table III are dependent only upon the selection of time intervals and upon laboratory data they are the same for all strain meters in any particular concrete structure.

Having the first five columns of Table III and a time-strain curve available, transcribe strains from the curve of Figure 6 to Column (6) at the corresponding ages given in Column (1). For greater ease in making that transcription and for later use in plotting stress, Column (11) may be filled by adding the corresponding ages from Column (1) to the initial date and time, that is, to the selected zero time for the particular strain meter. With Column (6) filled, the form is in readiness to begin the actual computation of stress.

Enter zeros on the first line of Columns (7) and (8) as those summations are zero at the beginning of the record. The value of $\Delta\sigma$ to be entered on the first line of Column (9) is the product of the values on the first line of Columns (5) and (6).

The value of $\Sigma f(K)\Delta\sigma$ to be entered on the second line of Column (7) is the product of the values on the first line of Column (9) and second line of Column (2).

The value of $\Sigma \frac{1}{E'} \Delta\sigma$ to be entered on the second line of Column (8) is the product of the value on the first line of Column (9) and the value on the second line of Column (3).

The value of $\Delta\sigma$ to be entered on the second line of Column (9) is obtained by the formula at the top of that column. All values so used are to be taken from the second line of the columns.

Symbolically, the computation of all succeeding values in Columns (7), (8), and (9) may be shown as follows:

Let numbers in parentheses represent columns.

Let subscripts to those numbers represent lines.

For example:

$(8)_4$ means the fourth line of Column (8).

$(7)_n$ means the nth line of Column (7).

To compute the nth line of Column (7):

$$(7)_n = (9)_{n-1}\ (2)_n + (7)_{n-1}$$

To compute the nth line of Column (8):

$$(8)_n = (9)_{n-1}\ (3)_n + (8)_{n-1}$$

To compute the nth line of Column (9):

$$(9)_n = (5)_n \left[(6)_n - (8)_n - (7)_n\ (4)_n\right]$$

From the foregoing it is evident that the values on any line in Columns (7), (8), and (9) must be computed before those in the succeeding line can be computed. It is not possible to compute any one or two of those columns independently.

Column (10) is the cumulative summation of Column (9).

SUMMARY

It is evident from a comparison of the stress curves of Figure 7 with the strain curve of Figure 6 that the greatest disagreement between the stress curves occurs during times when the strain is changing rapidly. At least a part, and possibly most, of the evident error is in the strain increment method and may be due to the use of strain increments rounded to whole numbers. It is not known whether Equation (6) and hence the average logarithm method is capable of rigorous mathematical proof. Some indeterminable error must be inherent in Equation (6) since its development was mainly by intuition and trial. However, it does make practicable the computation of stress records by punched-card calculating machines and is preferred by some to the strain increment method when desk calculators are employed.

It is believed that the errors incurred by the use of either method are within the limits of accuracy of the experimental data. The elastic and creep properties of concrete, too, are subject to such wide variations that high accuracy in any stress analysis of that material cannot be expected.

ACKNOWLEDGEMENTS

The development of the mathematical procedures described in this monograph is due to the efforts of several Bureau of Reclamation engineers and does not represent the work of the author alone. Among those who contributed to the origination and improvement of the methods were C. L. Townsend, Douglas McHenry, J. M. Raphael, J. A. Hanson, and J. R. Bruggeman.

The engineers of the Concrete Laboratory of the Bureau's Engineering Laboratories in Denver developed the techniques used in the reduction of the physical data to mathematical and graphical form.

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