SCOUR ANALYSIS UPSTREAM OF THE SAN ACACIA DIVERSION DAM ON THE RIO GRANDE

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Abstract In recent years, bank failure due to scour on the Rio Grande has been a recurring problem on a spoil levee upstream of the San Acacia Diversion Dam, threatening the safety of a nearby irrigation facility. In 2005 emergency work was needed to stabilize the eroding bankline, but this was considered only a temporary solution. A realistic estimate of the scour was needed at the project site for a long term solution,. While a number of empirical design equations were used to estimate the scour, the wide range of scour depth values made it difficult to determine a design scour depth. To resolve this issue, a two-dimensional (2D), depth-averaged, mobile-bed model was developed to predict the scour at the eroding bankline. Scour depths estimated by the 2D mobile-bed model scenarios compare reasonably with the scour depths estimated from empirical equations. The 2D mobile-bed model also identifies locations of maximum scour depth, which, when compared with field observations of the largest scour depth location, provided insight that helped refine and narrow the realistic scour depth range for this project. This study highlights the progress made with 2D mobile-bed models and how these models may be used to compute scour depths and locations.

INTRODUCTION

The right (north) bank of the Rio Grande, approximately 500 feet upstream of the San Acacia Diversion Dam and situated on a spoil levee protecting an irrigation facility, has experienced persistent bank erosion due to toe erosion. The erosion has been extensive enough in the recent past, spring 2005 for example, to warrant immediate maintenance repairs. The bankline section that is actively eroding is approximately located at River Mile (RM) 116.3 and is shown in Figure 1 as the Drain Unit 7 priority site location.

As part of the design process, a one-dimensional (1D) hydraulic model, HEC-RAS, was developed first to reflect the existing conditions and provide the variables necessary to proceed with a bankline protection design. As part of this process, a suite of empirical scour equations were evaluated to provide an estimate of the scour depth. The wide range of values estimated with these equations resulted in the implementation of a two-dimensional (2D) depth-averaged mobile-bed model. The SRH-2D model was used to predict the location and depth of the scour under a number of scenarios, further refining the process by which a design scour depth was determined.

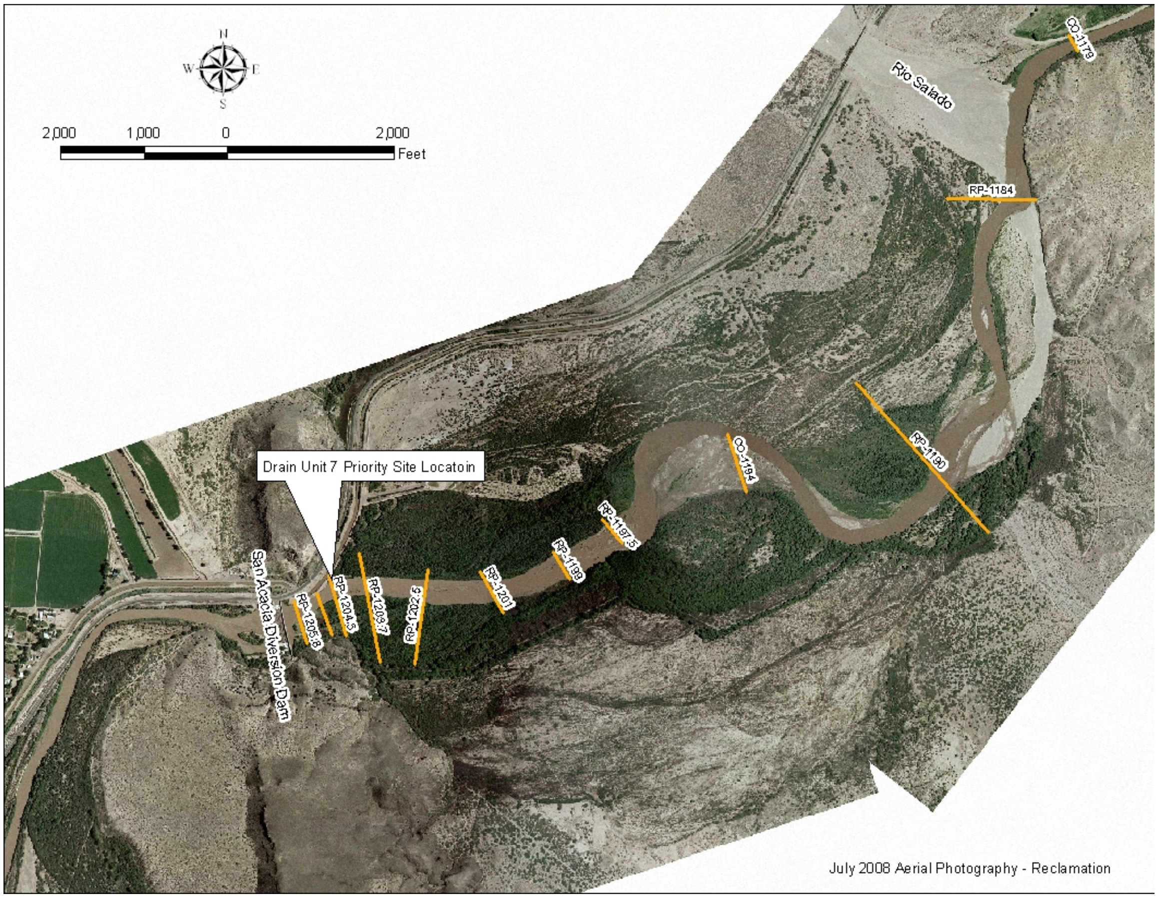


Figure 1. An aerial photo showing the bank erosion site

EMPIRICAL ANALYSIS

Scour results, based on empirical equations, were estimated at the design flow [16,450 cfs - 10 year return flow at San Acacia (Bullard and Lane, 1993)] using methods described in Pemberton and Lara (1984), Derrick and Freeman (2004), and Vanoni (2006). Two average bed size materials were used for these calculations (d_{50}): 0.40 mm and 2.7 mm. The sand sized materials (0.40 mm) were sampled on top of the river bed at the site in 2007, while the fine gravel materials (2.7 mm) were sampled upstream in 2000 (just downstream of the Rio Salado confluence). The fine gravel sediments were assumed to represent the armoring strata at the scour location, while the sand sized particles were assumed to represent the surface strata. A summary of the computed scour results are listed in Table 1. While the wide range of predicted values is to be expected given the different underlying assumptions and predictions of the empirical equations, it was desirable to further refine the scour depth predictions. The SRH-2D model was developed and used in order to gain a better understanding of the sediment processes at this project site and provide a more reliable estimate of the scour depth.

Table 1. Scour depth based on various empirical methods

	Method	Depth (ft) d_{50}=0.4 mm	Depth (ft) d_{50}=2.7 mm
General Scour for Moderate Bend (Pemberton and Lara, 1984)	Neill	6.0	6.0
	Lacey	5.8	4.2
	Blench	10.5	9.5
	Competent velocity	16.0	8.3
Bend Scour (Derrick and Freeman, 2004)	Thorne	11.0	11.0
	Maynard	9.1	9.1
	Zeller	3.5	3.5
	Apmann	5.1	5.1
Constriction Scour (Vanoni, 2006)	Straub	3.3	3.3
	Komura	3.5	3.5
	Griffith	3.5	3.5

A DESCRIPTION OF THE SRH-2D MODEL

Governing Equations SRH-2D is a 2D, depth-averaged, hydraulic and sediment transport model for river systems under development at the Bureau of Reclamation. A detailed presentation of the numerical method for the flow equations is omitted, and readers may refer to Lai (2006; 2010).

Sediment transport and mobile-bed dynamics are solved following the approach of Greimann et al. (2008) which divides non-uniform sediments into a number of sediment size classes (N_{sed}). Each size class *k* in the water column is governed by the following non-equilibrium mass conservation equation:

$$\frac{\partial hC_k}{\partial t} + \frac{\partial \cos(\alpha_k)\beta_k V_t hC_k}{\partial x} + \frac{\partial \sin(\alpha_k)\beta_k V_t hC_k}{\partial y} = \frac{\partial}{\partial x}\left(f_k D_x \frac{\partial hC_k}{\partial x}\right) + \frac{\partial}{\partial y}\left(f_k D_y \frac{\partial hC_k}{\partial y}\right) + \dot{V}_k \qquad (1)$$

In the above, subscript *k* denotes that the variable is for sediment size class *k*, C_k is the depth averaged sediment concentration by volume, $\beta_k = V_{sed,k} / V_t$ represents the sediment-to-flow velocity ratio, $V_t = \sqrt{U^2 + V^2}$ represents the depth-averaged total flow velocity, $V_{sed,k}$ is the concentration weighted, depth-averaged sediment velocity, α_k is the angle of the sediment transport direction relative to the *x*-axis, f_k is the “load” parameter representing the percentage of the sediments which would be in suspension (1.0 for the suspended load and 0.0 for the bed load), D_x and D_y are the mixing coefficients of sediments in the *x*- and *y*-directions, respectively, and $\dot{V}_k$ is the sediment exchange term between the water column and the channel bed. A number of parameters in equation 1 need to be defined; they include the sediment transport angle (α_k), the load parameter (f_k), the ratio of sediment-to-flow velocity (β_k), and the sediment exchange term ($\dot{V}_k$). Equations defining each of these variables may be found in Greimann et al. (2008) for the non-cohesive sediments.

Both cohesive and non-cohesive sediments are modeled in this study. The non-cohesive sediments are modeled as discussed by Greimann et al. (2008). The cohesive sediments are treated as a single special size class governed by equation 1. A different sediment exchange term, $\dot{V}_c$, instead of $\dot{V}_k$, is used as follows,

$$\dot{V}_c = V_e\, p_c - V_d\, C_c \qquad (2)$$

where V_e and V_d are the rate of erosion and deposition, respectively, and p_c is the percentage of the cohesive sediment on the bed. The erosion rate (V_e in mm/s) was derived from the measured data on the Rio Grande and is defined by:

$$V_e = 0 \qquad \text{for} \quad \tau \le \tau_{es} \qquad (3a)$$

$$V_e = S_s\left(\frac{\tau - \tau_{es}}{\tau_{em} - \tau_{es}}\right) \qquad \text{for} \quad \tau_{es} < \tau \le \tau_{em} \qquad (3b)$$

$$V_e = S_s + S_m\left(\frac{\tau - \tau_{em}}{\tau_{em}}\right) \qquad \text{for} \quad \tau > \tau_{em} \qquad (3c)$$

with $\tau_{es} = 0.125\, lb/ft^2$, $\tau_{em} = 2.84\, lb/ft^2$, $S_s = 0.25\, lb/hr \cdot ft^2$, and $S_m = 1.07\, lb/hr \cdot ft^2$. The deposition rate ($V_d$ in mm/s) is related to the fall velocity of the cohesive sediment and is based on the same measured data used to calculate the erosion rate. The defining equations for the deposition rate are listed in equation 4.

$$V_d = (1 - \frac{\tau}{\tau_{ref}})\omega \qquad \text{for} \quad \tau \le \tau_{df} \qquad (4a)$$

$$V_d = (1 - \frac{\tau}{\tau_{dp}})\omega(1 - \frac{C_{eq}}{C}) \qquad \text{for} \quad \tau_{df} < \tau < \tau_{dp} \text{ and } C > C_{eq} \qquad (4b)$$

$$V_d = 0 \qquad \text{for} \quad \tau \ge \tau_{dp} \text{ or } C \le C_{eq} \qquad (4c)$$

where $\tau_{ref} = \frac{\tau_{df}\tau_{dp}}{\chi\tau_{df} + (1-\chi)\tau_{dp}}$, $\chi = 1 - \frac{C_{eq}}{C}$, ω is the fall velocity, and the measured data gives $\tau_{df} = 0.005\, lb/ft^2$, $\tau_{dp} = 0.021\, lb/ft^2$, and $C_{eq} = 1.0\, g/l$.

The bed elevation (z_b) changes due to erosion and deposition within each size class. The change in z_b due to sediment size class *k*, $z_{b,k}$, is governed by equation 5.

$$\eta_{ak} \frac{\partial z_{b,k}}{\partial t} = -\dot{V}_k \tag{5}$$

where $\eta_{ak} = 1 - \sigma_{ak}$ is the porosity parameter and σ_{ak} is the porosity for the *k*-th size class in the active layer. The active layer is the top bed surface layer where sediment exchange occurs between the water column and the bed. Bed layers beneath the active layer, called subsurface layers, provide sediment to or receive sediment from the active layer. The volume fraction in the active layer is governed by the mass conservation equation, which is represented by equation 6.

$$\frac{\partial m_a p_{ak}}{\partial t} = -\dot{V}_k + p_{2k} \sum_i \dot{V}_i \qquad \text{for net erosion, } \sum_i \dot{V}_i \geq 0 \tag{6a}$$

$$\frac{\partial m_a p_{ak}}{\partial t} = -\dot{V}_k + p_{ak} \sum_i \dot{V}_i \qquad \text{for net deposition, } \sum_i \dot{V}_i < 0 \tag{6b}$$

where m_a is the total volume (or mass) of sediments in the active layer, p_{ak} is the volume fraction of *k*-th class in the active layer ($\sum_k p_{ak} = 1$), and p_{2k} is the volume fraction of *k*-th class in the first subsurface layer beneath the active layer.

The porosity of the active layer is governed by the volume conservation equation derived from the kinematic constraint and may be expressed by equation 7.

$$\frac{\partial \delta_{ak}}{\partial t} = -\frac{\dot{V}_k}{\tilde{\eta}_k} + p_{2k} \frac{\sum_i \dot{V}_i}{\eta_{2k}} \qquad \text{for } \sum_i \dot{V}_i \geq 0 \tag{7a}$$

$$\frac{\partial \delta_{ak}}{\partial t} = -\frac{\dot{V}_k}{\tilde{\eta}_k} + p_{ak} \frac{\sum_i \dot{V}_i}{\eta_{ak}} \qquad \text{for } \sum_i \dot{V}_i < 0 \tag{7b}$$

where $\tilde{\eta}_k$ is computed utilizing equation 8.

$$\tilde{\eta}_k = \eta_{ak} \ \text{ for } \dot{V}_k \geq 0 \ \text{ (}k\text{-th size is eroded from active layer)} \tag{8a}$$

$$\tilde{\eta}_k = \eta_{sk} \ \text{ for } \dot{V}_k < 0 \ \text{ (}k\text{-th size is deposited into active layer)} \tag{8b}$$

In the above, η_{sk} is the porosity parameter for the suspended sediment.

As the model iterates, the volume fraction (p_{Lk}), the porosity parameter (η_{Lk}), and the thickness (t_L) of subsurface layers are all updated. For this model, the subsurface layer underneath the active layer (layer 2) exchanges sediments with the active layer so that the mass of each size class is maintained in the active layer. In the process, the thickness of layer 2 may increase or decrease. The rest of the subsurface layers remain unchanged until the thickness of layer 2 is reduced to zero. When layer 2 has zero thickness, layer 3 assumes the role of layer 2.

Numerical Method The solution of the 2D, depth-averaged flow equations follows the method of Lai (2006; 2010). Basically all governing equations are solved using a finite volume method that ensures both a local and global mass conservation. The numerical methodology uses an unstructured hybrid mesh, following the methodology of Lai et al. (2003), employs an implicit time scheme with an automatic wetting-drying procedure, and adopts the segregated solution procedure utilizing the water surface elevation as the solution variable.

The sediment transport equation (1) is discretized similarly to the flow equations. The sediment "depth" hC_k is the main dependent variable and the fractional step method (Yanenko, 1971) is adopted as shown in equation 9.

$$\frac{(hC_k)^{\text{int}} - (hC_k)^n}{\Delta t} + \frac{\partial \cos(\alpha_k)\beta_k V_t (hC_k)^{\text{int}}}{\partial x} + \frac{\partial \sin(\alpha_k)\beta_k V_t (hC_k)^{\text{int}}}{\partial y} + \frac{\partial}{\partial x}\left(f_k D_x \frac{\partial (hC_k)^{\text{int}}}{\partial x} \right) + \frac{\partial}{\partial y}\left(f_k D_y \frac{\partial (hC_k)^{\text{int}}}{\partial y} \right) = 0 \quad (9a)$$

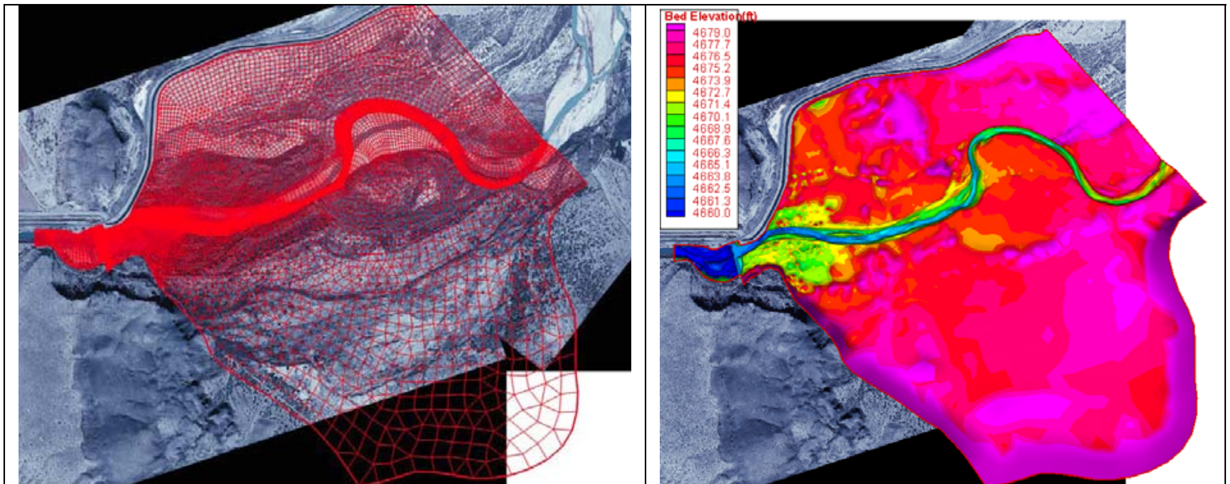
$$\frac{(hC_k)^{n+1} - (hC_k)^{\text{int}}}{\Delta t} = \dot{V}_k \quad (9b)$$

The advection equation (9a) is solved implicitly to obtain intermediate solutions $(hC_k)^{\text{int}}$ with known values $(hC_k)^n$ at time level *n*; the initial value problem in equation (9b) is solved analytically to obtain the new solution $(hC_k)^{n+1}$ at time level (*n+1*).

A decoupled solution procedure between the flow and sediment equations is adopted. Within each time step, an iterative solution is obtained for the flow equations using known results at the old time level *n*. Water surface elevation and flow velocity values (hereinafter referred to as the flow variables) are thus obtained at the new time level *(n+1)*, assuming that sediment concentration and bed elevation are known at time *n*. The sediment concentration and bed elevation are then solved based on the flow variables at time level *(n+1)*.

2D MODELING OF THE SCOUR

Model Description A large solution domain was chosen (Fig.2a) to predict the scour (Lai and Bauer, 2007). The upstream boundary is located at about 1.8 miles from the San Acacia Diversion Dam and the downstream boundary is approximately 1120 feet downstream of the Dam. The lateral dimension was created to be wide enough to contain the 25-year flood. The mesh used consists of both quadrilaterals and triangles with a total of 12,595 cells and 11,640 points (see Fig.2a). Elevation data for this model was compiled from a number of sources. River bathymetric data for the Rio Grande was collected in 2007 by Reclamation using an Acoustic Doppler Current Profiler, ADCP (Bauer, 2007). Floodplain topography was based on USGS DEMs, LiDAR data from 1999, photogrametrically derived cross section data obtained in 2002, and topographical surveys conducted in 2003. A compilation of this data, shown in figure 2b, was used to create the surface from which the mesh point elevation was interpolated.



(a) The solution domain and mesh | (b) Bed elevation contours

Figure 2. The solution domain, mesh and topography for the simulation

The Manning's coefficient (*n*) for bed roughness (flow modeling) and the gradation for representing the surface and subsurface bed sediments (scour and mobile-bed analysis) are needed for the entire solution domain. In this study, the solution domain was divided into twelve bed-type zones; in each zone the Manning's coefficient (*n*) and bed gradation were assigned (Lai and Bauer, 2007). The Manning's coefficients were based on a previous modeling study for this project site utilizing a 1D, HEC-RAS, model that used a main channel value of 0.026 and an overbank value ranging from 0.04 to 0.06. The bed gradation data was obtained from bed samples collected at the site in 2007 during this study.

The scour area was represented by a single bed-type zone. Two bed gradations were used in this zone: the cohesive-bed and the non-cohesive-bed. With the cohesive-bed, two bed layers were used: the surface layer and the subsurface layer. The surface layer had a one foot depth and was composed of sands and gravels with a gradation consistent with field measurements in the project area (see Table 2 –cohesive bed top sediment). The subsurface layer consisted of cohesive materials with erodibility properties as described in section 3.1. It was assumed with the cohesive-bed condition that this subsurface layer had an infinite thickness. It is possible, based on observations elsewhere on the Rio Grande, that multiple, alternating layers of cohesive and non-cohesive sediments exist, but the assumption of an infinitely thick cohesive bed provides a lower end estimate of the scour depth, as cohesive sediment is very resistance to bed erosion. This provides a means of bracketing the possible scour that may occur at the project site.

The non-cohesive bed represents the other extreme of the bed materials and was intended to provide a high end estimate of the scour depth, giving the other bracket for possible scour at the project site. For the non-cohesive condition the bed was assumed to have an infinite thickness and consist of sand sized particles with a d_{50}=0.4 mm (see Table 2- non-cohesive).

Table 2. Measured bed gradations at two locations that are used for modeling

Bed Condition	64 mm	32 mm	16 mm	8 mm	4 mm	2 mm	1 mm	0.5 mm	0.25 mm	.125 mm	.063 mm	.004 mm
Cohesive-bed top sediment	100	96.0	85.2	66.0	51.2	42.9	38.5	30.1	12.3	3.30	1.10	0
Non-cohesive	100	98	92.4	82.6	74.9	70.3	67.3	59.6	24.	6.85	1.65	0

Eight size classes (see table 3) were used to represent the nonuniformity of the sediments at the project site. Size class 1 represents cohesive sediment while size class 8 represents non erodible bedrock. Since the existing bank is lined with riprap the bankline at this location was modeled as non erodible.

Table 3. Sediment diameter bounds of each size class for the modeling

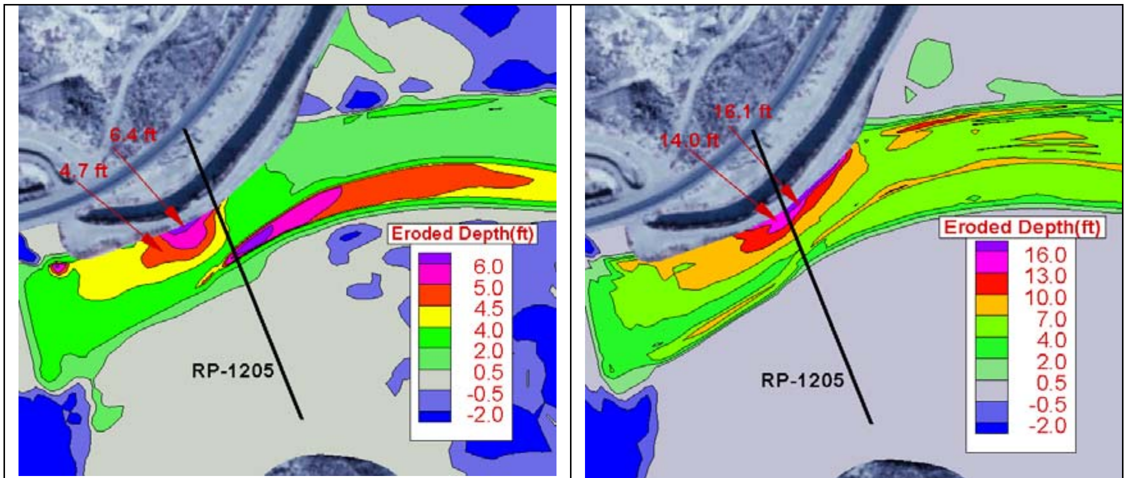
Size Class No.	1	2	3	4	5	6	7	8
Lower d(mm)	<.0625	.0625	0.25	1.0	2.0	8.0	32.0	rock
Upper d(mm)		0.25	1.0	2.0	8.0	32.0	125.	

Boundary Conditions and Other Model Parameters Boundary conditions at both the upstream and downstream model boundaries are needed. For the upstream both the discharge and sediment supply rate were specified. The discharge (16,400 cfs) was based on the 10 year return flow at San Acacia (Bullard and Lane, 1993). The sediment supply rate was assumed to equal the transport capacity for the non-cohesive sediments and the equilibrium concentration ($C_{eq} = 1.0\, {}^{g}\!/\!{}_{l}$) for the cohesive sediments. Since the upstream boundary is located significantly upstream of the project site location, this transport capacity assumption was considered adequate.

For the downstream, the water surface elevation from a previous HEC-RAS modeling for this project site was applied. Since flow is supercritical across the exit of the San Acacia Diversion Dam, the downstream boundary condition is less critical.

Other modeling parameters include the bulk dry density for the non-cohesive sediment (99.26 lb/ft^3), the bulk dry density for the suspended cohesive sediment (30.0 lb/ft^3), and the bulk dry density for the bed cohesive sediment (58.0 lb/ft^3). These parameters were based on previous modeling efforts, SRH-1D, on the Rio Grande at other locales.

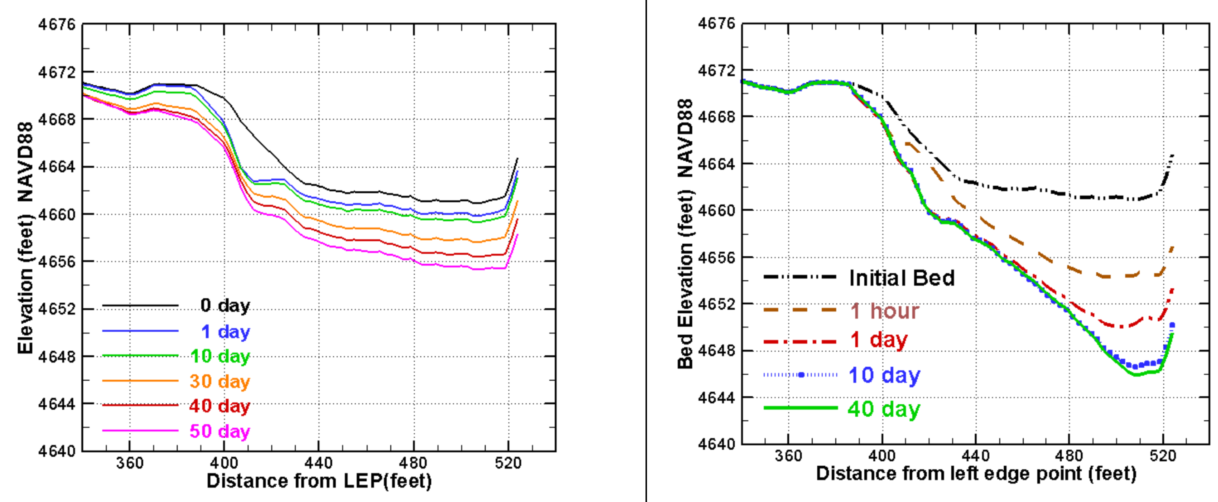
Results and Discussion The unsteady, mobile-bed simulations were run for 40 days. The predicted scour at day 40 for both the cohesive and the non-cohesive bed conditions are shown in Figure 3. Observations of the maps in this figure reveal that the maximum scour is near cross section line RP-1205, being upstream of the line for the non-cohesive bed and downstream of the line for the cohesive bed. With the cohesive-bed, the scour depth is about 6.6 ft and the maximum scour is located about 76 ft downstream of the RP-1205 line. With the non-cohesive-bed, the maximum scour depth predicted is 19.2 ft and is located about 43 feet upstream of the RP-1205 line. Field measurements made at the site showed that the deepest scour is located upstream of the RP-1205 line. It seems likely, therefore, that the actual bed condition is probably closer to the non-cohesive bed condition, though the two bed scour estimates providr a probable bracket for the design scour.



(a) Cohesive-bed (b) Non-cohesive-bed

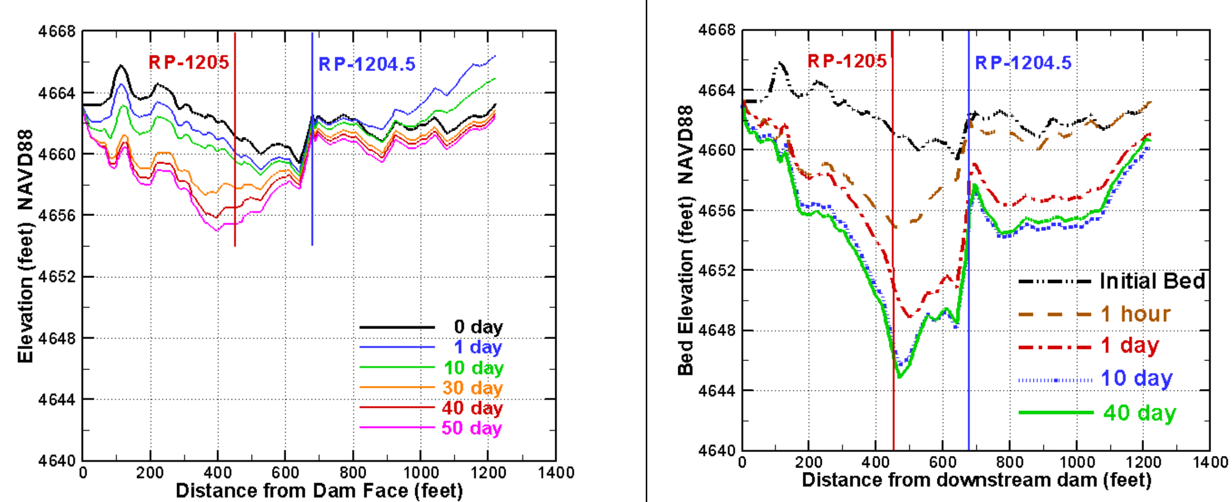
Figure 3. Predicted scour depth after 40 days (positive for net erosion and negative for deposition).

The time evolution of the scouring process can be viewed by plotting the bed elevation along two lines: one on the RP-1205 line, shown in Figure 3 and another along the toe of the right bank. Figures 4 and 5 show the bed elevation changes with time along the two lines for the two scenarios. As mentioned previously, the equilibrium maximum scour depth is not attained for the cohesive bed, as the scour process is slow and a much longer time than 40 days is required. Continued bed scour is expected for the cohesive bed condition until the bed shear stress is below the critical erosion stress. For the cohesive bed condition, the surface erosion critical stress is 0.125 lb/ft^2 (or 4.0 $lbm/ft/s^2$) and the mass erosion critical stress is 2.84 lb/ft^2 (or 91.3 $lbm/ft/s^2$). The bed shear stress predicted at day 40 for the cohesive bed is still above 100 $lbm/ft/s^2$ (Lai and Bauer, 2007), which is significantly higher than the surface erosion critical stress and slightly higher than the mass erosion critical stress. In contrast, the scouring process for the non-cohesive-bed is much faster. For example, the scour reached a depth of 10 ft between RP-1205 and RP-1204.5 after 6 hours. As the depth of scour increases the scour process slows down significantly and reaches equilibrium after a few days. The graphs in figure 4 and 5 show that the difference in scour between 10 and 40 days is small.



(a) Cohesive-bed (b) Non-cohesive-bed

Figure 4. Predicted bed elevation change along the RP-1205 line



(a) Cohesive-bed (b) Non-cohesive-bed

Figure 5. Predicted bed elevation change along the toe of the right bank

CONCLUDING REMARKS

A 2D, depth-averaged, mobile-bed, numerical model was developed to simulate the scouring process at an outer bend upstream of the San Acacia Diversion Dam on the Rio Grande. The purpose was to gain a better understanding of the scouring process and obtain a more reliable estimate of the scour at the project site. The predicted scour from the 2D model range from a value of 6.6 ft in the cohesive sediment to 19.2 feet in the non-cohesive sediment after a 40 day simulated run. Empirical equations were also used to estimate the scour at this site and provide scour component estimates ranging from 3.1 to 16.0 feet, implying a total scour estimate within the range predicted by the 2D modeling effort. The benefit of the 2D modeling effort was that the results are based on field site conditions, as opposed to trying to match the site conditions upon which the empirical equations are based. The ability of the 2D model to predict the actual location of the maximum scour, also helped, when compared to field data measurements of the actual maximum scour depth, to flush out which bed scenario was more pertinent to the project site, thereby further helping to choose a design scour depth. This study shows that, while there is still a level of uncertainty with predicting scour depth, 2D mobile-bed models have been advanced in recent years to such a point that they may be used to predict the scouring process and help delineate the design scour with reasonable confidence.

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