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THEORETICAL CONSIDERATIONS ON DISCHARGE MEASUREMENTS BY THE ALLEN METHOD

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THEORETICAL CONSIDERATIONS ON DISCHARGE MEASUREMENTS BY THE ALLEN METHOD

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A translation of

Considerations theoriques sur la mesure des debits d'eau par la methode d'Allen

by

P. de Haller

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Translated by

Edward F. Wilsey, Assistant Engineer
U. S. Bureau of Reclamation

Denver, Colorado

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Summary

Measurements of the discharge of water by the Allen method are affected by errors arising from the principle of the method itself. The importance of different sources of errors and means of attenuating their effects has been investigated on the basis of the theory of turbulent flow.

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A new method for the determination of the discharge of water, conceived by the American engineer, Charles M. Allen, has been frequently used for some years. It is chiefly applicable to measurements in canals and pressure conduits; the measurement is simple and rapid and does not necessitate a previous calibration nor experimental coefficients. However, it is not absolutely free of errors arising from the principle itself; the aim of this work is a purely theoretical study of the method with a view to evaluating the possible errors independent of the errors in observation and those arising from the imperfections on the apparatus. It appears possible to reduce these errors considerably by modifying the apparatus. This seems most desirable considering the numerous advantages of a practical nature inherent in the Allen method.

The principle is as follows:[1,2]

(1) A concentrated solution of salt is injected over a short interval of time at the upper extremity of a conduit. This solution modifies the electrical conductivity of the water. (2) Pairs of electrodes, each pair being connected to a source of current and to a recording galvanometer, are placed at any two sections of a conduit. Each pair of electrodes detects the passage of the salt solution. The time, T, separating these two passages and also the volume, V, of the conduit between the electrodes is measured. The discharge is evidently:

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1 Allen, C. M. and Taylor, E. A.: The Salt-Velocity Method of Water Measurement; Trans. American Society of Mechanical Engineers, vol. 45, 1923.

2 Müller: Die Salzgescwhindigkeits-Methode von Allen; Schweizerische Bauzeitung; vol. 87, Jan. 23, 1926.

$$Q = \frac{V}{T}$$

However, the solution forms a cloud which occupies a certain volume within the conduit. The deflections of the galvanometers will not be instantaneous, but will occupy a certain time; hence the determination of T presents an uncertainty.

Allen overcame this difficulty in the following fashion: Let A (figure 1) be the curve recorded by a galvanometer during the passage of the solution past the first pair of electrodes and let B be a similar curve for the second pair of electrodes. The centers of gravity T_1 and T_2 of the two cross-hatched areas are determined; the time separating these two points will be the time, T, to introduce into the formula.

This procedure is justified. Briefly, in order to determine the discharge, it is necessary to determine the average velocity of the water. However, the average velocity of a deformable body is equal to the velocity of its center of gravity. It will suffice therefore to determine the instants that the center of gravity of the cloud formed by the solution passes by the electrodes at 1 and at 2 (figure 2).

Suppose now that an ideal pair of electrodes is placed at a section whose abscissa is x_1. These electrodes will occupy the entire section of the conduits and their conductivity is constant at all points. Such electrodes can be conceived as two trellises made up of very thin conductors. These trellises are very close to one another and in no way disturb the free flow of the liquid. They are insulated electrically from one another and are subjected to a constant difference of potential. At the passage of the cloud formed by the solution, a current is set up between each pair of electrodes. If the concentration and consequently the conductivity of the solution is constant, this current will be directly proportional to the area, S, of the cloud at the plane of each pair of electrodes, or

$$i = \alpha S$$

Suppose in addition that during a passage by a pair of electrodes, the cloud is not deformed, that is to say, the velocity, v, at every point of the cloud is the same. In order to simplify the reasoning, it may be assumed that the cloud is stationary and that the first pair of electrodes is displaced with a velocity, v. Then

$$x - X = v(t - T_1) \; ; \quad dx = v\, dt$$

where X_1 = the abscissa of the center of gravity of the cloud. Also

$$\int (x - X_1) S \, dx = 0$$

Combining we have

$$\frac{1}{\alpha} \int (x - X_1) i \, dx = 0 = \frac{V^2}{\alpha} \int i (t - T_1) dt$$

or

$$\int i (t - T_1) dt = 0$$

This shows clearly that T_1 is the abscissa of the center of gravity of the curve recorded by the galvanometer.

The same reasoning applies to the second pair of electrodes at the section whose abscissa is X_2. Allen's procedure is therefore correct, providing the following three conditions are satisfied:

(1) The velocity of every particle constituting the cloud remains constant. In other words, the cloud is not deformed between the two pairs of electrodes.

(2) The electrodes are nets which occupy the entire section of the conduit and do not disturb the flow of water; they have a constant conductivity at every point.

(3) The concentration is constant at every point of the cloud.

Practically none of these three conditions is satisfied; the first, because the particles in the neighborhood of the walls of the conduit are held back, and, in consequence, the velocity of flow in this region is different from that which obtains at the center of the conduit; the second, because such electrodes cannot possibly be made and, hence, present considerable resistance to the flow; and the third, because, even though it were possible to obtain a fairly constant concentration of the cloud at the moment of injection, diffusion will quickly modify this condition.

Therefore, there are three sources of error independent of one another and all of which can influence the results of measurements.

We shall now attempt to estimate their effect.

First of all, let us assume that conditions (2) and (3) are satisfied and attempt to evaluate the ~~error~~ arising from the deformation of the cloud. This deformation is due to the fact that the velocity of flow is not a constant at every point on a diameter of the conduit. We know that in the case of laminar flow, the distribution of velocity is parabolic, zero at the walls, and maximum at the center. Turbulent flow in pressure conduit differs from laminar flow in that the path of a fluid particle is no longer a straight line but an irregular line oscillating back and forth in all directions without, in the meantime, going astray very far. Diffusion and the viscosity both increase in intensity; the velocities of flow that are used in calculations are not the instantaneous values but are the mean values depending on a certain interval of time. The general principles of laminar flow are also applicable to turbulent flow. Only the numerical constants differ. In particular, the distribution of velocity on a diameter of the conduit is no longer a parabola; it is generally admitted[1] that it may be represented by the formula:

$$v = u_0 \sqrt[n]{1 - \frac{r^2}{R^2}} \tag{1}$$

where u_0 = the velocity at the center of the conduit and n is an exponent which varies within certain limits depending on the local conditions and especially on Reynolds' Number, $Ⓡ = \frac{UD}{\nu}$, ν being the kinematic viscosity,

Consider now a cylindrical pipe whose diameter is $D = 2R$. A salt solution is injected at point $x = 0$ and at the time $t = 0$ (fig. 3). The solution first forms a cloud bounded by two parallel planes at a distance, e, apart; let v be the velocity of any point P on the front of the cloud at a distance r from the axis of the pipe. This front has a tendency to deform because of the difference of velocity between adjacent points on the front. At some instant, t, the displacement of P can be represented by the equation:

$$x = vt$$

1 Handbuch der Physik (Geiger und Scheel) vol. 7, p. 144, J. Springer, Berlin, 1927
Karman, T. v; Zeitschrift für angewandte Mathematik und Mechanik, vol. 1, 1921, p. 238.

where v has the value indicated by equation (1); introducing this, we have

$$x = u_0 t \sqrt[n]{1 - \frac{r^2}{R^2}}$$

The plane at X_1 cuts the front of the cloud in the following circle of radius r':

$$r'^2 = R^2 \left[1 - \left(\frac{X_1}{u_0 t} \right)^n \right] \qquad (2)$$

It cuts the back of the cloud in a circle of radius r" or

$$r''^2 = R^2 \left[1 - \left(\frac{X_1 + e}{u_0 t} \right)^n \right] \qquad (3)$$

According to the assumed conditions, the current passing between a pair of electrodes is simply proportional to the area of the cloud at the plane X_1 or $i = \alpha S$. S is evidently:

$$S = \pi \left(r'^2 - r''^2 \right)$$

Therefore the equation for the curve recorded by the galvanometer is

$$i = \alpha \pi R^2 \left[\left(1 - \left(\frac{X_1}{u_0 t} \right)^n \right) - \left(1 - \left(\frac{X_1 + e}{u_0 t} \right)^n \right) \right]$$

The abscissa of the center of gravity, T_1, is defined as follows:

$$T_1 = \frac{\int i\, t\, dt}{\int i\, dt}$$

$$\int i t\,dt = c_1 \int_{t=\frac{X_1}{u_0}}^{\infty} \left(1 - \left(\frac{X_1}{u_0 t}\right)^n\right) t\,dt \; - c_1 \int_{t=\frac{X_1+e}{u_0}}^{\infty} \left(1 - \left(\frac{X_1+e}{u_0 t}\right)^n\right) t\,dt \;*$$

$$= - c_1 \left[\left(\frac{X_1+e}{u_0}\right)^2 - \left(\frac{X_1}{u_0}\right)^2\right] \frac{n}{2(n-2)}$$

and

$$\int i\,dt = c_1 \int_{t=\frac{X_1}{u_0}}^{\infty} \left(1 - \left(\frac{X_1}{u_0 t}\right)^n\right) dt - c_1 \int_{t=\frac{X_1+e}{u_0}}^{\infty} \left(1 - \left(\frac{X_1+e}{u_0 t}\right)^n\right) dt$$

$$= - c_1 \left(\frac{X_1+e}{u_0} - \frac{X_1}{u_0}\right) \frac{n}{n-1}$$

Therefore

$$T_1 = \left(X_1 + \frac{e}{2}\right) \frac{n-1}{n-2} \, \frac{1}{u_0} \qquad (5)$$

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* Translator's note.

The integration may be performed as follows: Change the limits of the first integral, thus

$$\int_{X_1/u_0}^{\infty}\left(1-\left(\frac{X_1}{u_0 t}\right)^n\right)t\,dt = \int_{\frac{X_1+e}{u_0}}^{\infty}(1-\left(\frac{X_1}{u_0 t}\right)^n t\,dt + \int_{X_1/u_0}^{\frac{X_1+e}{u_0}}\left(1-\left(\frac{X_1}{u_0 t}\right)^n\right)t\,dt$$

We now have three integrals. Combine the two having the same limits and rewrite with the third integral, thus

$$\int_{\frac{X_1+e}{u_0}}^{\infty}\left[-\left(\frac{X_1}{u_0}\right)^n+\left(\frac{X_1+e}{u_0}\right)^n\right]\frac{t\,dt}{t^n}+\int_{X_1/u_0}^{\frac{X_1+e}{u_0}}\left[1-\left(\frac{X_1}{u_0}\right)^n\left(\frac{1}{t^n}\right)\right]t\,dt$$

Integrating and substituting the limits we have

$$-\left(\frac{X_1}{u_0}\right)^n\frac{1}{(n-2)\left(\frac{X_1+e}{u_0}\right)^{n-2}}+\left(\frac{X_1+e}{u_0}\right)^n\frac{1}{(n-2)\left(\frac{X_1+e}{u_0}\right)^{n-2}}+\frac{1}{2}\left(\frac{X_1+e}{u_0}\right)^2$$

$$-\frac{1}{2}\left(\frac{X_1}{u_0}\right)^2+\left(\frac{X_1}{u_0}\right)^n\frac{1}{(n-2)\left(\frac{X_1+e}{u_0}\right)^{n-2}}-\left(\frac{X_1}{u_0}\right)^n\frac{1}{(n-2)\left(\frac{X_1}{u_0}\right)^{n-2}}$$

Simplifying, we have

$$\left(\frac{X_1+e}{u_0}\right)^2\frac{1}{n-2}+\frac{1}{2}\left(\frac{X_1+e}{u_0}\right)^2-\left(\frac{X_1}{u_0}\right)^2\frac{1}{n-2}-\frac{1}{2}\left(\frac{X_1}{u_0}\right)^2$$

Further simplification gives

$$\left(\frac{X_1+e}{u_0}\right)^2\left(\frac{1}{2}+\frac{1}{n-2}\right)-\left(\frac{X_1}{u_0}\right)^2\left(\frac{1}{2}+\frac{1}{n-2}\right)$$

The integration of $\int i\,dt$ is analogous and results in

$$\left(\frac{X_1+e}{u_0} - \frac{X_1}{u_0}\right)\left(1+\frac{1}{n-1}\right)$$

An analogous calculation for the second electrode at the abscissa X_2 gives:

$$T_2 = \left(X_2 + \frac{e}{2}\right)\frac{n-1}{n-2}\,\frac{1}{u_0} \tag{5a}$$

Therefore the average velocity of flow, U_a, corresponding to that measured by the Allen method will be:

$$U_a = \frac{X_2 - X_1}{T_2 - T_1} = \frac{n-2}{n-1}\,u_0 \tag{6}$$

The true mean velocity, U, deduced from the law of the distribution of velocity is

$$U = u_0\,\frac{n}{n+1} \qquad (7)^*$$

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* Translator's note.

The average velocity is computed from

$$U = \frac{\int_0^R u_0\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{n}} 2\pi r\,dr}{\pi R^2}$$

$$= -u_o \int_0^R \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{n}} d\left(1 - \frac{r^2}{R^2}\right)$$

$$= -u_o \left[\frac{\left(1 - \frac{r^2}{R^2}\right)^{\frac{n+1}{n}}}{\frac{n+1}{n}} \right]_0^R = u_o\left(\frac{n}{n+1}\right)$$

The relative error defined by

$$\varepsilon = \frac{U_a - U}{U}$$

is

$$\varepsilon = \frac{n-2}{n-1}\,\frac{n+1}{n} - 1 = -\frac{2}{n(n-1)} \qquad (8)$$

This error is negative, that is to say, the method of Allen will give results that are too small. The error rises rapidly to 0 for increasing values of n. For n = 7, which is a frequent mean, this error is minus 4.8 percent (fig. 4).

For an n less than 2, formula (8) ceases to be valid. This arises from the fact that the curve i is a function of t. Referring to figure 5 it is seen that for values of n less than 7, the curve flattens out so much that the integral

$$\int i\,t\,dt$$

ceases to have a meaning for n $\leqq 2$. As a result, we have the paradox that for laminar flow corresponding to n = 1, according to equation (8) Allen's method is not applicable. In practice this is not the case. Actually, we are obliged to insulate electrically the electrodes from the walls of the pipe; therefore, the electrodes do not occupy all of the area of the conduit and therefore do not record the passage of the slowly moving particles in the immediate neighborhood of the walls. It is precisely these particles which are the cause of the spreading out of the curve of the current for small values of n. We shall now study in more detail the design of electrodes.

Take the case where condition 2 is no longer satisfied. The electrode will have such a form that it is symmetrical with respect to the axis of the pipe. Its contribution to the current is therefore a function of the radius r.

Let di be the current passing between corresponding elements of area, dS, on a pair of electrodes. Referring to figure 6, we shall have then

$$di = \alpha\, dS$$

α and dS both being functions of r. The total current i will be expressed by:

$$i = \int_{r''}^{r'} \alpha\, dS = F(r') - F(r'') \qquad (9)$$

r' and r" have here the same meaning as in the preceding discussion. But with a redistribution of velocity r' is uniquely a function of $\frac{X_1}{u_o t}$ and r" of $\frac{X_1+e}{u_o t}$. The equation of the curve recorded by the galvonometer is therefore

$$i = \psi\left(\frac{X_1}{u_o t}\right) - \psi\left(\frac{X_1+e}{u_o t}\right)$$

or i is equal to the difference of the ordinates of the two curves,

$$y = \psi\left(\frac{X_1}{u_o t}\right) \; ; \quad y_1 = \psi\left(\frac{X_1+e}{u_o t}\right)$$

The abscissa of the center of gravity of the curve of the current can be written as follows:

$$T_1 = \frac{\int \left(t + \frac{e_1}{2}\right) e_1 \, dy}{\int e_1 \, dy} \; ; \text{ where } \; y = \psi \left(\frac{X_1}{u_o t} \right)$$

e_1 as it is easily seen, is proportional to t or

$$e_1 = e \frac{t}{X_1}$$

Thus:

$$\int \left(t + \frac{e_1}{2}\right) e_1 \, dy = \frac{e}{X_1} \left[1 + \frac{e}{2X_1} \right] \int t^2 \, dy$$

and

$$\int e_1 \, dy = \frac{e}{X_1} \int t \, dy$$

Consequently:

$$T_1 = \left[1 + \frac{e}{2X_1} \right] \frac{\int t^2 \, dy}{\int t \, dy}$$

The two integrals have the limits:

$$t = \frac{X_1}{u_o} \quad \text{and} \quad t = \infty$$

when the electrodes occupy the entire diameter of the pipe. Since

$$y = \psi\left(\frac{X_1}{u_0 t}\right)$$

we can change the variable; thus:

$$\frac{X_1}{u_0 t} = Z$$

Therefore,

$$T_1 = \left[X_1 + \frac{e}{2}\right] \frac{1}{u_0} \frac{\int_1^0 \frac{1}{Z^2} \psi'(Z) dZ}{\int_1^0 \frac{1}{Z} \psi'(Z) dZ} \tag{10}$$

Analogously for T_2 with

$$Z = \frac{X_2}{u_0 t}$$

we have:

$$T_2 = \left[X_2 + \frac{e}{2}\right] \frac{1}{u_0} \frac{\int_1^0 \frac{1}{Z^2} \psi'(Z) dZ}{\int_1^0 \frac{1}{Z} \psi'(Z) dZ} \tag{10a}$$

Since the limits of the integrals in 10 and 10a are the same, we can put

$$K = \frac{\int_1^\infty \frac{1}{Z^2}\psi'(Z)dZ}{\int_1^\infty \frac{1}{Z}\psi'(Z)dZ}$$

and hence

$$T_1 - T_2 = (X_2 - X_1)\frac{K}{u_0}$$

The average velocity U_a is:

$$U_a = \frac{X_2 - X_1}{T_2 - T_1} = \frac{u_0}{K} \tag{12}$$

This mean velocity does not depend solely on the form of the electrodes and on the distribution of velocity, both of these quantities being contained in the factor K, but also on the location of the electrodes in the conduits.

One type of electrode frequently used is shown in figure 8: Two iron plates of width b are placed a short distance apart and along a diameter of the pipe, their distance apart diminishing linearly from the center of the pipe to the walls.

The element of current di is here, approximately

$$di = \alpha \frac{b}{a} dr$$

and since

$$a_1 = a_0 - \beta r \quad ; \quad \beta = \frac{a_0 - a_1}{R}$$

then

$$i = \alpha \int_{r''}^{r'} b \frac{dr}{a_0 - \beta r} = -A \left| \log (a_0 - \beta r) \right|_{r''}^{r'}$$

By making the same assumptions as heretofore concerning the distribution of velocity, we have

$$r' = R\sqrt{1-\left(\frac{X_1}{u_0 t}\right)^n}$$

and

$$i \sim \log\left[\frac{a_0}{a_0-a_1}-\sqrt{1-\left(\frac{X_1}{u_0 t}\right)^n}\right]-\log\left[\frac{a_0}{a_0-a_1}-\sqrt{1-\left(\frac{X_1+e}{u_0 t}\right)^n}\right] \quad (13)$$

The function ψ in this case is

$$\psi(z) = \log\left[\frac{a_0}{a_0-a_1}-\sqrt{1-\left(\frac{X_1}{u_0 t}\right)^n}\right]$$

and, therefore:

$$\varepsilon = \frac{U_a - U}{U} = \frac{n+1}{kn} - 1 \quad (14)$$

in which K is computed as defined above.

We have determined this coefficient, K, and the corresponding error graphically for different values of n and for $a_0/a_1 = 5$. The error is still negative and invariably greater than that obtained with ideal electrodes as defined by condition 2 (fig. 4). However, for small values of the ratio a_0/a_1 the error is small. It increases for increasing values of $\frac{a_0}{a_1}$ as is shown in figure 9 which is plotted for n = 7.

Let us now take the general case where the electrodes do not occupy the entire diameter of the pipe. This means that for radii, r, larger than a certain radius, r_a, the contribution to the current is 0.

Integral 9 for the value of the current i, will therefore have the following limits:

r'' and r' if $r' < r_a$

r'' and r_a if $r' \geqq r_a$

This means that in figure 7 curves y and y_1 will not pass beyond the ordinate y_a. It is necessary to determine the center of gravity of the area ABCD. This calculation is entirely analogous to the one made above. The coefficient K is now defined by

$$K = \frac{\int_1^{Z_a} \frac{1}{Z^2} \psi'(Z)\,dZ}{\int_1^{Z_a} \frac{1}{Z} \psi'(Z)\,dZ} \tag{15}$$

Where Z_a is obtained from the relation

$$Z_a = \sqrt[n]{1 - \left(\frac{r_a}{R}\right)^2}$$

Let us apply this to the ideal electrodes for which

$$\psi(Z) = 1 - Z^n \quad ; \quad \psi'(Z) = -n Z^{n-1}$$

$$K = \frac{n-1}{n-2} \frac{1 - Z_a^{n-2}}{1 - Z_a^{n-1}} \tag{16}$$

For $r_a = R$, $Z_a = 0$, and

$$K = \frac{n-1}{n-2}$$

Using this value in equation (14) will result in an expression which agrees with equation (8).

For $r_a = 0$, $Z_a = 1$, $K = 1$, and $U_a = u_0$.

Figure 10 shows the error ε as a function of r_a for $n = 7$, for both ideal and ordinary electrodes with $\frac{a_0}{a_1} = 5$.

Starting from $r_a/R = 1$, it is seen that, at first, the error decreases rapidly. It passes through 0 when $r_a = 0.96\ R$, then becomes positive and attains its maximum value for $r_a = 0$ when the electrodes are reduced to points and the velocity obtained is evidently that which obtains at this point, that is to say, u_0.

This suggests a study of the case where this point electrode instead of being placed at the center of the pipe is placed at a distance r_p from the axis. Let us in particular search for the distance r_p for which the error is 0.

The nature of the curve recorded by the galvonometer is now different. At the moment when the front of the cloud reaches the electrode, the current passes rapidly to a certain value, remains constant during the passing and falls rapidly to 0. The asymptotic tail is therefore eliminated. Such a tail makes the determination of the center of gravity difficult (fig. 14). On condition that r_p is the same for the two pairs of electrodes, we have simply:

$$v(T_2 - T_1) = X_2 - X_1 \quad \text{where} \quad U_a = v$$

and since

$$U = \frac{n}{n+1} u_0 \; ; \; v = u_0 \sqrt[n]{1 - \frac{r^2}{R^2}}$$

in order that the error be 0, it is necessary that $U_a = U$, or

$$U_a = v = \frac{n}{n+1} u_0 = u_0 \sqrt[n]{1 - \frac{r^2}{R^2}} \; ; \; \frac{r_p}{R} = \sqrt{1 - \left(\frac{n}{n+1}\right)^n} \qquad (17)$$

This distance r_p varies very little relative to n (fig. 11). It may therefore be expected that given a zero error for n = 7, the error for a value of r_p remains small if the distribution of velocity is different from that defined for n = 7. Figure 12 shows that this error is less than 1 percent for all values of n greater than n = 4; it is insignificant for values of n greater than 7.

We have assumed up to now that the concentration was the same at all points of the cloud formed by the solution.

In fact, even though a constant concentration is realized at the moment of the injection of the solution, diffusion will tend to disburse the cloud so that its boundary will be less and less well defined. It may be foreseen that a first effect of this diffusion will be to round the sharp angles of the theoretical curve recorded by the galvanometer. Thus, it approaches nearer to the curve recorded experimentally.

Diffusion in a regime of turbulent flow is not simply a phenomenom that is purely molecular as it is in the case of a fluid at rest. The particles are transported by an irregular movement (turbulent) which has a considerably greater amplitude than molecular motion. Their special properties, electrical conductivity, temperature, and momentum are also changed. It is therefore the same mechanism which governs diffusion that also governs the thermal conductivity, the viscosity, etc.[1]

Let us imagine now a quantity G somehow attached to each particle and whose value varies from one point to another. The amount of the quantity G passing per unit of time across a unit of area will be

$$\frac{\partial G}{\partial t} = \delta \frac{\partial G}{\partial x}$$

$\frac{\partial G}{\partial x}$ being the gradient of the variation of G along the normal to the element of area, and the coefficient of diffusion, δ , being a quantity independent of the nature of G.

If G is the momentum ρv, $\frac{\partial \rho v}{\partial t}$ is the tangential force by definition equal to $\tau = \eta \frac{\partial v}{\partial x}$ where η is the coefficient of viscosity. From this: $\rho \delta = \eta$; $\delta = \frac{\eta}{\rho}$

1 Handbuch der Physik, vol VII, p. 138

We arrive, therefore, at the important result that the coefficient of diffusion is equal to the ratio of the coefficient of viscosity to the density of the fluid. The coefficient of viscosity, η, is not the same as that obtained by viscosimeter measurements and may be several times greater. It is an apparent coefficient analogous to the "turbulence" introduced by Boussinesq.

In a circular pipe, the tangential force, τ, is not constant at every point of a transverse section; it may be easily shown that it is proportional to the radius; this is a consequence of the fact that the pressure should be constant on all planes perpendicular to the axis of the pipe. We have:

$$\tau = \tau_o \frac{r}{R} = \eta \frac{\partial v}{\partial r}$$

Assuming that the distribution of velocity can be expressed by the equation

$$v = u_o \sqrt[7]{1 - \left(\frac{r}{R}\right)^2}$$

we have

$$\frac{\partial v}{\partial r} = \frac{2}{7} u_o \frac{r}{R^2} \left(1 - \frac{r^2}{R^2}\right)^{-6/7}$$

from which we obtain

$$\eta = \frac{7}{2} \frac{\tau_o R}{u_o} \left(1 - \frac{r^2}{R^2}\right)^{6/7}$$

The loss of pressure per unit of length is

$$\Delta p = \lambda \frac{\rho U^2}{2D}$$

or again,

$$\Delta p = \tau_0 \frac{4}{D}$$

where U is the mean velocity of flow and D the diameter of the conduit. Blasius[1] gives the following expression for λ:

$$\lambda = 0.316 \sqrt[4]{\frac{1}{Ⓡ}} \; ; \; Ⓡ = \text{Reynolds' Number}$$

The coefficient of diffusion is computed to be,

$$\delta = 0.0605 \; \nu \, Ⓡ^{3/4} \left(1 - \frac{r^2}{R^2}\right)^{6/7}$$

Roughly, Reynolds' Number is of the order of 10^6, hence

$$\delta = \sim 1910 \, \nu \left(1 - \frac{r^2}{R^2}\right)^{6/7}$$

This is more than 2,000 times the value of the coefficient of diffusion for still water.

The fact that δ is a function of $\frac{r}{R}$ makes it possible to find a general solution of the problem. However, in order to obtain some idea of the importance of diffusion, we assume that it does act only along the axis of the pipe, thus we neglect diffusion along a radius. This radial diffusion is probably small because at the center of the pipe where the coefficient of diffusion is large, the gradient of the concentration of the solution is small, in consequence of the symmetry; and, inversely, in the neighborhood of the walls where the concentration gradient is high, the coefficient of diffusion is small.

Assuming that the diffusion is purely axial, the equation of the problem is

1 Handbuch der Physik, vol. VII, p. 140

$$\frac{\partial c}{\partial t} = \delta \frac{\partial^2 c}{\partial x^2} \tag{19}$$

and its general solution can be written in the form[1]

$$c = \frac{1}{2\sqrt{\delta}\sqrt{\pi t}} \int_0^{\infty} f(s) \left[e^{\frac{-(x-s)^2}{4\delta t}} + e^{\frac{-(x+s)^2}{4\delta t}} \right] ds \tag{20}$$

$c = f(x)$ being the redistribution of the concentration at the time, $t = 0$, the moment of injection. At this instant, the cloud is bounded by two parallel planes at a distance, e, apart. Therefore $f(x)$ is as follows:

for $-e/2 < x < +e/2$, $f(x) = c_0$
for $x < -e/2$ and $x > -e/2$, $f(x) = 0$

and equation (20) becomes:

$$C = \frac{c_0}{2} \left[\phi \left[\frac{e/2 - x}{2\sqrt{\delta t}} \right] + \phi \left[\frac{e/2 + x}{2\sqrt{\delta t}} \right] \right] \tag{21}$$

where ϕ represents the integral of Gauss' error[2], or

$$\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-z^2} dz$$

1 Courant and Hilbert: Methoden der mathematischen Physik (Methods of Mathematical Physics) I, p. 61, J. Springer, Berlin.
2 Jahnke and Emde: Funktionentafeln mit Formeln und Kurven (Tables of Functions with Formulas and Curves), B. G. Teubner, Leipzig, 1928.

By introducing into this equation the value of δ calculated above and by placing $x = vt$ in order to take into account the deformation of the cloud, we can calculate the concentration at each instant at each point of the pipe and from this it is easy to determine the current passing between a pair of electrodes as a function of the time.

It is hardly possible to draw general conclusions here. We shall confine ourselves to the computation of a numerical example given the following:

Diameter of the pipe = D = 5.5 meters (18.0 feet)
Mean velocity = U = 3.1 meters per second (10.2 feet per second)
Time of injection = 0.97 seconds
Lengths of the cloud = e = 3 meters (9.8 feet)

The first pair of electrodes is placed at 17 meters (55.8 feet), the second at 54 meters (177 feet) from the section at which the solution is injected.

The coefficient of diffusion is:

$$\delta = \delta_o\left(1 - \frac{r^2}{R^2}\right)^{6/7}$$

where

$$\delta_o = 0.0605\,\nu\,(R)^{3/4}\ ;\ \nu = 0.01$$

$$(R) = 1.705 \times 10^7\ ;\quad \delta_o = 160.5\ \text{cm}^2/\text{sec.}$$

Figure 14 shows the curve of the current such as would be recorded by a point electrode placed at a distance $r_p = 0.8R$ from the axis of the pipe. It is seen that the diffusion serves to round the angles and to elongate the curve without, however, displacing appreciably the center of gravity.

In figure 15 are shown curves such as would be recorded by electrodes occupying the whole section of the pipe. Here again, the diffusion does not have a large influence. Calculations based on these two curves, thus taking into account the diffusion, show a velocity of flow whose mean value is 2.96 meters per second (9.71 feet per second) which is different by 0.2 percent from that predicted by the elementary theory, and this difference is, moreover, less than the precision of calculation.

In spite of the high value of the coefficient of diffusion in this particular case, the diffusion does not modify sensibly the numerical results obtained heretofore. On the contrary, it represents the true phenomenon much more exactly, as is shown by the third curve of figure 15, which is taken from a recording of which figure 16 is an exact copy. This record has been obtained in the course of tests on a pipe which satisfies approximately the conditions which we have admitted; in regard to the theoretical curve, even though it is not absolutely identical to that which we could hope to expect, at least it shows a similarity of character which leads us to think that our reasoning is valid.

Conclusions.

As a basis for this study, we have assumed that the apparatus necessary for applying the Allen method, functions in an irreproachable and theoretically perfect manner. This being admitted, we have shown that the various errors which may occur depend essentially on the velocity distribution along a diameter of a conduit and the form of the electrodes. It is theoretically possible, given a law for the velocity distribution, to choose a form of electrode such that the error is 0. But if the anticipated distribution is not realized, an error reappears. We arrive, therefore, at the conclusion that in order to make reliable measurements, it is necessary to know this distribution of velocity.

The form of electrode which gives theoretically the best results, is a point electrode placed at a distance from 0.75R to 0.8R from the axis of the conduit; as we have seen, large differences in the velocity distribution occasion only a minimum error, on condition, however, that the flow is perfectly symmetrical.

Some very careful measurements made on a conduit of a large power plant have shown that this is not generally the case. The section of measurement was in the middle reach of a straight pipe 220 meters (722 feet) long. Consequently, it was very favorably situated. Measurements had been made, using Dufour's current meter method, at 38 points distributed along two diameters perpendicular to each other. The distribution of velocity was found to be unsymmetrical, more so along a horizontal diameter than along a vertical diameter. The results of four measurements made at the same distance from the axis on both horizontal and vertical diameters, differed from each other by more than 4 percent; nevertheless, their mean is almost exactly the same as that given by formula (1) which we have used. n will have the following values corresponding to the velocity of flow at the center of the conduits, u_0:

u_0 m./sec.	n	R
0.81	5.65	1.51×10^6
1.39	5.80	2.61×10^6
2.22	6.50	4.23×10^6
2.56	6.75	4.91×10^6

With such an unsymmetrical velocity distribution the Allen method, utilizing point electrodes, will give very uncertain results. It will be preferable, therefore, to use annular electrodes whose radius is 0.77R (fig. 17). The velocity which will be measured will not be exactly the average velocity at each point of this electrode, but an approximate value which, in this particular case, will differ by only 0.3 percent.

We have shown that even by assuming the most favorable conditions as to the uniformity of the flow, the perfect functioning of the apparatus, etc., Allen's method leads to results which are subject to errors, whose magnitude, however, is rarely possible to predict but which can be expected to amount to several percent. As regards the flow phenomena, we have used the minimum of arbitrary assumptions. It may be objected that the initial form of the cloud is not as simple as we have assumed; in this case, however, our reasoning leads to the same general results. The numerical calculations given have only the value of an example; it will be perfectly illusory to ask them to serve as correct experimental results.

Small scale laboratory tests have shown a very good agreement between measurements by the Allen method and weir measurements. It is therefore probable that the sources of error that we have pointed out are not the only ones; and it is possible that the different errors neutralize each other.

Whatever the case may be, this method does not appear to be above all criticism. Certainly it should be possible to find some arrangement of and some form of electrode such that the error is reduced to a minimum. Furthermore, the apparatus actually in use is susceptible to considerable improvement. On the other hand, the Allen method possesses the more essential advantages. Most important of all, it permits the making of a larger number of measurements in a short period of time. Thus, even in its present form, it is of great service each time that it will suffice to know the discharge to a few percent. Only a large number of comparable tests covering a wide range of conditions will disclose whether or not it is also applicable to cases where a high degree of precision is required.

We wish to thank the Chief Engineer, Ackeret, for the numerous suggestions that he has made in the course of the elaboration of this work.

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FIGURES
X-D-238D
X-D-238B
X-D-238

THEORETICAL ELECTRODE DESIGNS

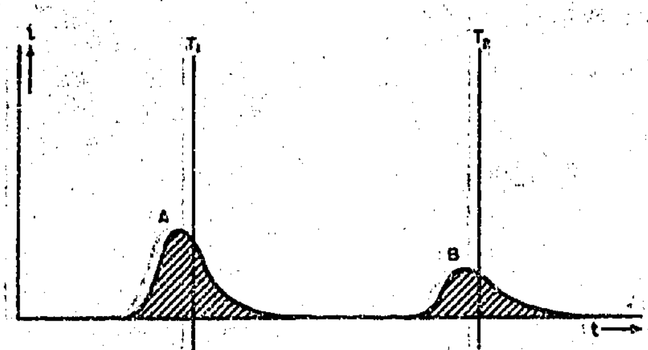

FIGURE 1

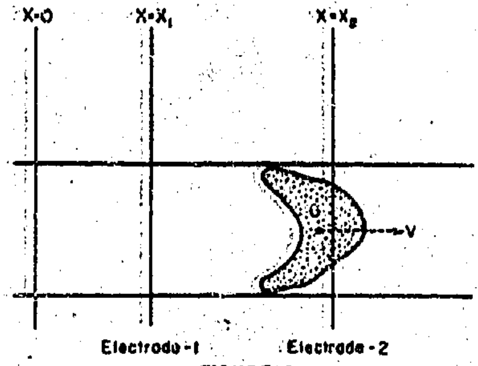

FIGURE 2

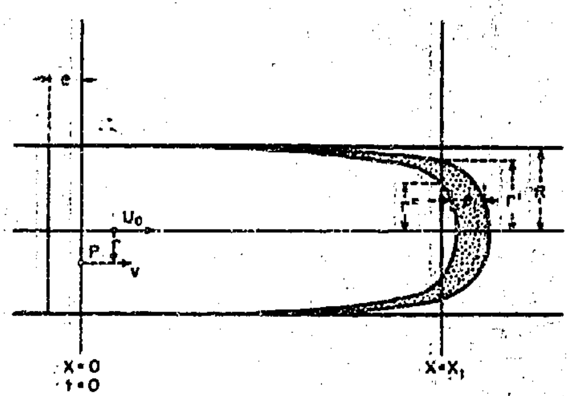

FIGURE 3

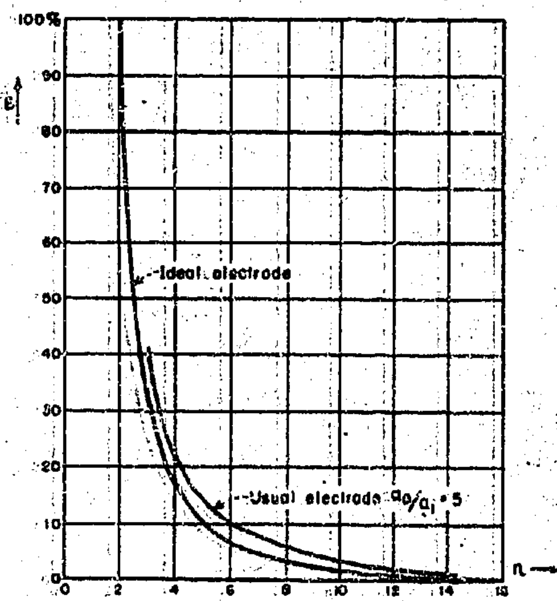

FIGURE 4 — THE ERROR AS A FUNCTION OF n

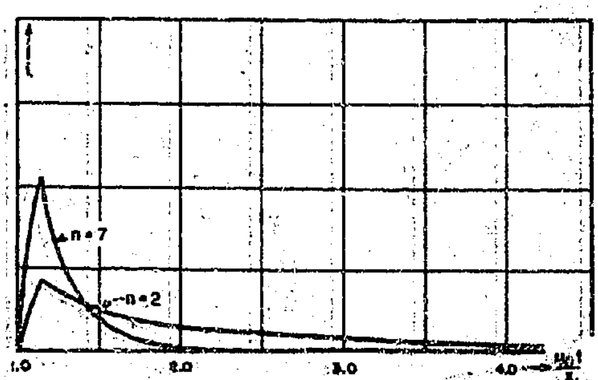

FIGURE 5 — CURVES OF THE CURRENT AS A FUNCTION OF TIME

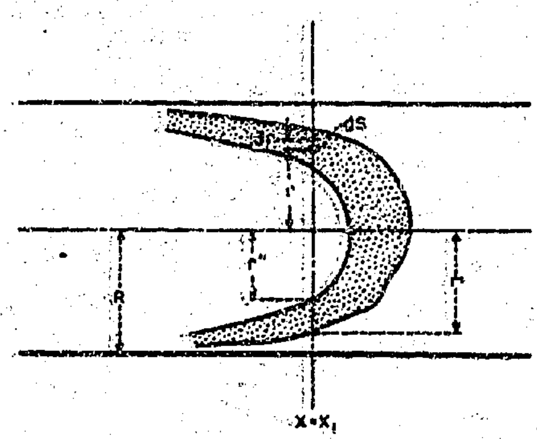

FIGURE 6

THEORETICAL ELECTRODE DESIGNS

HYD-33

X-D-2350

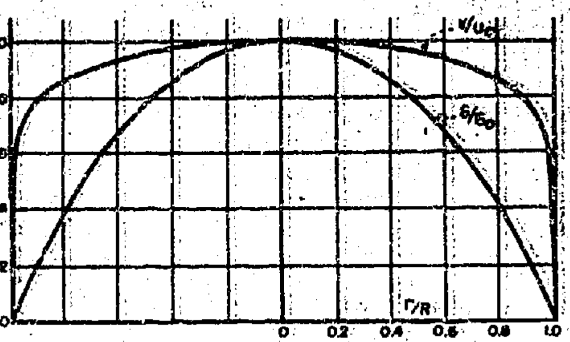

FIGURE 13 – DISTRIBUTION OF VELOCITY AND OF THE COEFFICIENT OF DIFFUSION

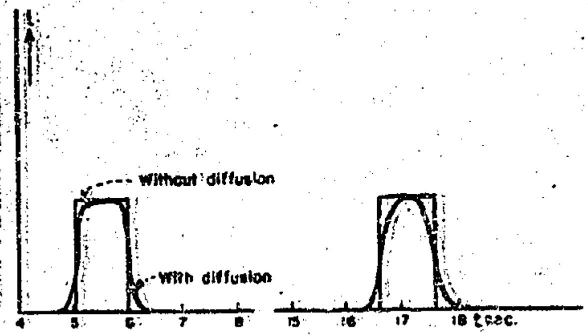

FIGURE 14 – CURVE OF THE CURRENT FOR A POINT ELECTRODE

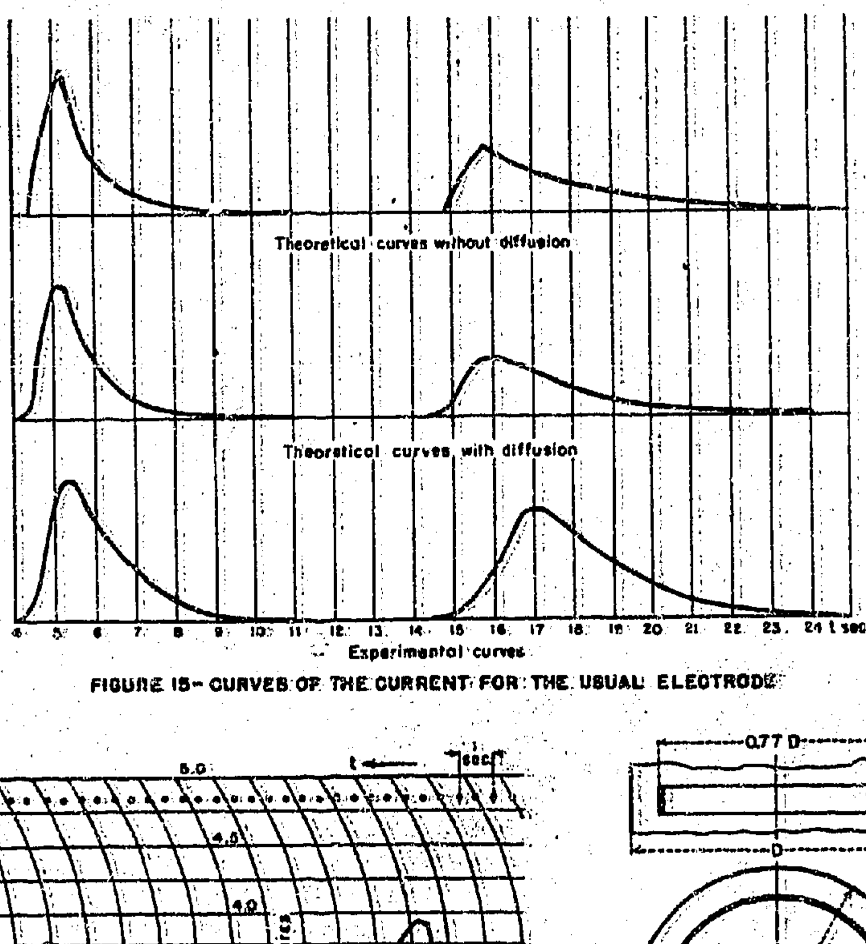

FIGURE 15 – CURVES OF THE CURRENT FOR THE USUAL ELECTRODE

FIGURE 16 – FAC-SIMILE OF A RECORDING FROM A GALVANOMETER

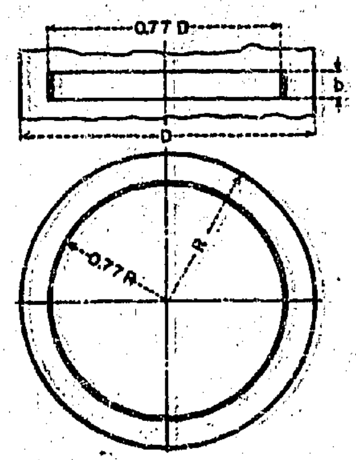

FIGURE 17 – AN ELECTRODE FOR REDUCING THE ERROR

THEORETICAL ELECTRODE DESIGNS

HYD-33

X-D-2382

THEORETICAL ELECTRODE DESIGNS

THEORETICAL ELECTRODE DESIGNS

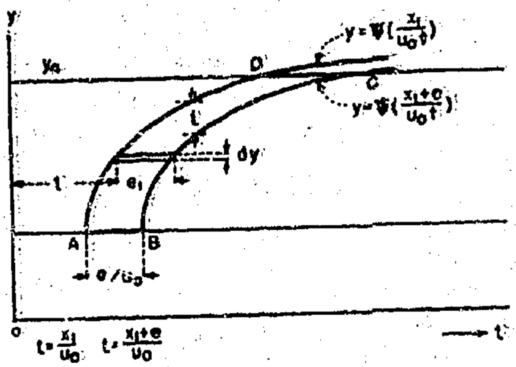

FIGURE 7

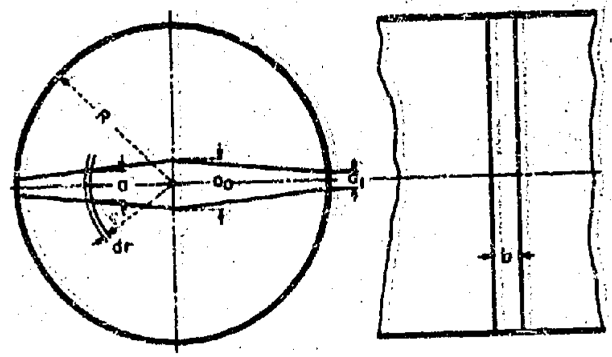

FIGURE 8 - THE USUAL ELECTRODE

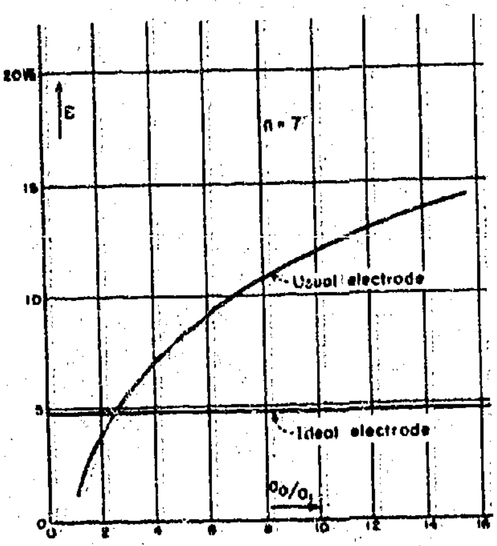

FIGURE 9 - ERRORS FOR THE TWO TYPES OF ELECTRODES

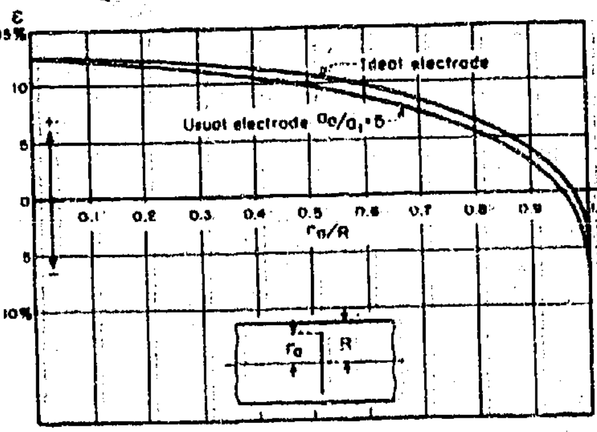

FIGURE 10 - ERROR FOR AN ELECTRODE NOT EXTENDING ALONG THE ENTIRE DIAMETER OF A CONDUIT

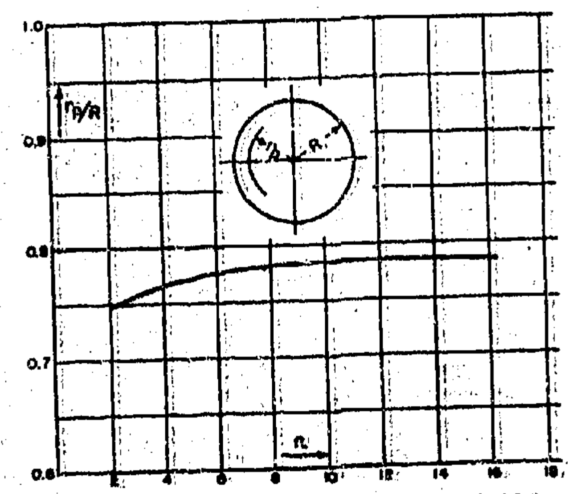

FIGURE 11 - LOCATION OF A POINT ELECTRODE FOR A ZERO ERROR

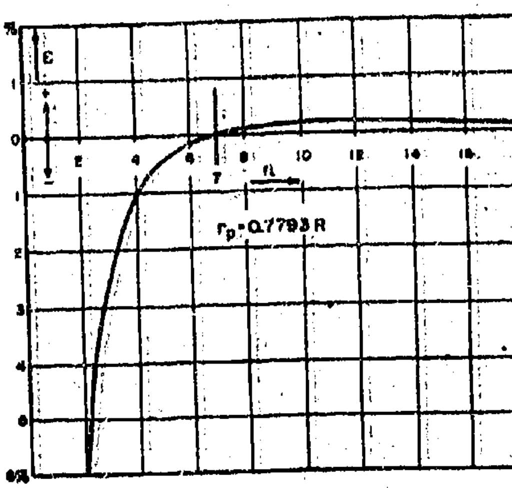

FIGURE 12 - THE ERROR AS A FUNCTION OF THE EXPONENT n FOR A POINT ELECTRODE

THEORETICAL ELECTRODE DESIGNS

HYD-331

X-D-2781