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Vacuumless Dam Profiles

by

A. Vitols

Translated by

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See also

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Model Research at Spillway Crests

VACUUMLESS DAM PROFILES

By A. Vitols

A translation of

"Beitrag zur Frage des vacuumlosen Dammprofiles"

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TRANSLATOR'S PREFACE.

This paper is of special interest due to the fact that Vitols has introduced a term for the effect of the centrifugal force on the overfalling nappe.

A discussion of the integration of the differential equation by A. Fischer and the rebuttal to this discussion by A. Vitols appeared in Wasserkraft und Wasserwirtschaft, vol. 32, 1937, p. 71. It did not appear worthwhile to add a translation of this argument to the translation of the original paper.

Edward F. Wilsey

Denver,
June 1, 1937.

The unfavorable action of negative pressures on spillways creates the following three conditions:

1. A negative pressure reduces the back pressure on the spillway so that the resultant pressure is correspondingly increased;

2. A negative pressure causes, as a result of the vibrations accompanying it, a localized disintegration of the surface material of the dam, for the nappe first adheres then springs free from the wetted surface of the spillway;

3. The entire dam can acquire a state of considerable vibration; the displacements of a single point on the dam are indeed not large; the succeeding impulses, however, by increasing the acceleration and by inaugurating forces within the body of the dam can give rise to important secondary forces of varying significance. This phenomenon often increases to such an extent that it resembles an earthquake in the vicinity of the power plant. One can well imagine how destructive such a play of force can be on the whole structure. In this connection, there are plenty of examples of projects of faulty construction.

The vacuum problem has aroused great interest in recent times, particularly in countries where an active demand for water power has arisen as, for example, in the Soviet Union, where an attempt to solve this problem has resulted in a report of the Hydrotechnical Institute of Leningrad[1].

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[1] "Bulletin de l'Institut Hydrologique", edited by L. S. Berg; Leningrad, 1927; paper in Russian by Madame L. V. Kasanskaya, pages 80 to 120.

The most popular solution of the problem appears to be that of the American engineer, Creager[2]. It, as well as other investigations, is based on the weir experiments of Bazin[3] concerning the shape

[2]"Engineering for Masonry Dams", New York, 1917, p. 105.

[3]Annales des Ponts et Chaussees, vol. 6, no. 19, 1890, 1st semester, p. 56; Also a German translation by K. Keller, Zeits, des V.D.I., vol. 34, 1890, p. 883.

of an aerated free nappe at the upper portion of its trajectory. Briefly, Creager's procedure is as follows: A particle of water at a distance of about one-third of the thickness of the nappe measured from its under surface (figure 2) is conceived as a free solid particle which moves in a vacuum, that is, without friction. The initial location of this particle corresponds to the point of maximum pressure in the nappe (figure 1) and the initial velocity of the particle is the average over the whole section. The trajectory of the particle is the parabola:

$$x^2 = \frac{2V^2_{ox}\, y}{g}$$

where V_{ox} is the horizontal component of the initial velocity.

If this curve is plotted on Greater coordinate paper and if the origin is shifted, the following Creager equation results:

$$(x-a)^2 = \frac{2V^2_{ox}}{g}\,(y + b) = 2.732\,(y + b)$$

The constants a, b, and V_{ox} are determined by experiment for a unit head. This equation represents the kern between the upper and lower nappe profiles. The thickness of the nappe at any point along its trajectory is determined as follows: The vertical component of the velocity is:

$$V_y = \sqrt{2gy}$$

the horizontal component,

$$V_{ox} = \text{const.}$$

and the resultant velocity is,

$$V = \sqrt{V_{ox}^2 + V_y^2}$$

The thickness of the jet is found from,

$$D = \frac{q}{V}$$

The kern lies at a distance of D/3 measured orthogonally from the under surface of the nappe. Proceeding in this fashion, both the upper and lower profiles may be plotted. The Creager table of coordinates is calculated for h = 1; for other heads, the coordinates are increased in proportion to the head.

Madame Kasanskaya, a Russian hydraulic engineer, has criticized Creager's procedure; she questions the possibility of finding the constants a and b by experiment, and objects to the point chosen by Creager as the free point.

In her analysis, she has chosen the center of mass of the nappe as the free point.

The geometrical interpretation of this proposal is that the free point is the centroid of the velocity surface as calculated from its first moment. She has calculated this free point and has found it to be at 0.444 of the thickness of the nappe from the under the trajectory:

$$y = n + mx + kx^2$$

The constants n, m, and k are determined from Bazin's experiments by applying the theory of least squares to the observed coordinates.

The coordinates of the nappe profiles are computed in the same way as by Creager, that is, from the thickness of the jet and the path of the free particle.

The proposal of the Russian, Pusirevsky, should also be mentioned, and again is concerned with the motion of a free particle. He also introduces a pressure force which influences the motion of the free particle.

It is to be noted that in this case the ordinary parabola described by a free fall is distorted. He gives no fixed method, for determining this force, and, therefore, there is danger of distorting the parabola either too much or too little. The details of this procedure cannot be reproduced here; the reader is referred to the original paper.

It must be said that the above methods all proceed from the concept of a "free particle", with which hydraulics has little or nothing to do. Apparently, a solution can be achieved on the basis of ordinary hydraulics; otherwise, all propositions of this kind cannot rise above the level of rank empiricalism and thus can render reliable results only in certain cases where they accurately fit experiments.

The unreliability of an empirical procedure is increased whenever one departs from the conditions of the experiments. In the following, an attempt is made to solve the vacuum problem on the basis of the everyday hydraulics assisted by Bazin's experiments.

The Analysis

The problem is reduced to the special case of a dam with a vertical upstream face, (figure 3), which, however, is found frequently in practice. Naturally this analysis can be extended to other cases.

The well-known Bernoulli equation is expanded to include an additional term for the purpose of introducing into the discussion the centrifugal force which accompanies curvilinear motion and which reduces the pressure between the nappe and the surface of the spillway.

The expanded equation is as follows:

$$\frac{\nu_0 V_0^2}{2g} + H = H_0 = \frac{\nu V^2}{2g\varphi^2} - y + \int_0^s \cos\alpha' \, ds - \frac{1}{g}\int_0^s \frac{V'^2}{r'} \, ds \qquad (1)$$

in which:

V = the average velocity of the nappe;

V_0 = its initial value;

ν = the coefficient of Coriolis;*

ν_0 = the initial value of this coefficient;

*Translator's note: ν is usually denoted by American writers as α, and is defined by the following equation:

$$\nu \frac{V^2}{2g} = \frac{\nu (Q/A)^2}{2g} = \frac{\gamma \int_0^A (v')^3 \, da}{2g\gamma \int_0^A v' \, da} = \frac{\int (v')^3 \, da}{2gQ}$$

This coefficient might well be called Coriolis' coefficient, as in this paper.

ϕ = the velocity coefficient;

α' = the angle between the vertical and the tangent at a point on the curve, s, which is drawn orthogonally to the stream lines;

v' = the local velocity of a particle of water tangent at any instant to a stream line; and

r' = the radius of curvature of a stream line at the position of the particle.

The coefficient ϕ comes from the expression for the energy loss in the following manner: The sum of the velocity head at any point in the nappe and the loss of head is:

$$\frac{\gamma V^2}{2g} + \varsigma \frac{\gamma V^2}{2g} = \gamma (1 + \varsigma) \frac{V^2}{2g} = \frac{\gamma V^2}{2g} \cdot \frac{1}{\phi^2}$$

The geometrical symbols are shown in figure 3.

The meaning of the term $\int_0^s \cos \alpha' \, ds$ is as follows (see figure 4): The increment of hydrostatic pressure is given by,

$$dp = \frac{dp}{ds}\, ds = \bar{\gamma} \cdot \overline{ds} \text{ (scalar product)}$$

where γ = the specific weight.

This expression may be written as follows:

$$\left|\bar{\gamma} \cdot \overline{ds}\right| = \gamma\, ds \cos(\bar{\gamma}\, \overline{ds}) = \gamma\, ds \cos \alpha' = \gamma\, ds'$$

and $$\int_0^s dp = p = \gamma \int_0^s \cos \alpha'\, ds = \gamma \int_0^{s'} ds' = \gamma s'$$

$$= \gamma \beta h' = \beta \gamma h \cos \alpha$$

It follows from these equations that:

$$\int_0^s \cos\alpha' \, ds = \beta h \cos\alpha = \frac{p}{\gamma}$$

where β = a constant.

We now put:

$$\int_0^s \frac{(v')^2}{r'} ds = \frac{V^2}{r} \mu' s = \frac{V^2}{r} \mu h$$

in which:

μ = a constant;

r = the radius of curvature of the spillway profile wetted by the nappe; and

V = the average velocity of the jet, as defined by the following relation,

$$AV = Q = \int^A v' \, dA = b \int_0^s v' \, ds$$

where A = the orthogonal cross section of the nappe.

Bernoulli's equation now has the following form:

$$(H_0 + y) = \frac{\nu V^2}{\phi^2 2g} + h\left(\beta \cos\alpha - \frac{\mu V^2}{r g}\right)$$

$$= \frac{\nu}{\phi^2} \frac{V^2}{2g} + \Pi h$$

where;

$$\Pi h = h\left(\beta \cos\alpha - \frac{\mu V^2}{r g}\right) = h \varepsilon$$

$h\varepsilon$ is the piezometric height at the boundary between the nappe and the spillway. We now assume the limiting condition that the nappe is always in contact (the under side of the nappe is identical with the spillway profile) when the piezometer height is zero or;

$$\Pi = \varepsilon = \beta \cos\alpha - \frac{\mu v^2}{2rg} = 0$$

We now have the parametric equations:

$$H_o + y = \frac{\nu}{\phi^2}\frac{v^2}{2g} \qquad (2)$$

and

$$\beta \cos\alpha - \frac{\mu v^2}{2rg} = 0 \qquad (3)$$

These equations determine the limiting profile of a vacuumless spillway for the above condition. The variables in these equations are y, v, α, and r and the constants β, ν, ϕ, and μ may be replaced by a single new constant or:

$$K = \frac{\mu \phi^2}{\nu \beta}$$

We now proceed to eliminate v from equations (2) and (3), thus:

From (2),

$$\frac{v^2}{g} = 2(H_o + y)\frac{\phi^2}{\nu}$$

substituting for $\frac{v^2}{g}$ in (3), we obtain;

$$\beta \cos\alpha - \frac{\mu \phi^2}{\nu}\left(\frac{H_0 + y}{r}\right) = 0$$

or $$1 - \frac{\mu \phi^2}{\nu \beta} \cdot \frac{H_0 + y}{r \cos\alpha} = 1 - \frac{K(H_0 + y)}{r \cos\alpha} = 0$$

where $K = \dfrac{\mu \phi^2}{\nu \beta}$

Our next task is to write this equation in terms of y and its derivatives, thus:

$$r = \frac{(1 + y'^2)^{3/2}}{y''}$$

and $$\tan\alpha = \frac{dy}{dx} = y' = \frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{1 - \cos^2\alpha}}{\cos\alpha}$$

or, $$y'^2 \cos^2\alpha = 1 - \cos^2\alpha$$

or $$\cos\alpha = (1 + y'^2)^{-1/2}$$

Therefore:

$$r \cos\alpha = \frac{(1 + y'^2)^{3/2}}{y''}(1 + y'^2)^{-1/2} = \frac{1 + y'^2}{y''}$$

and $$1 - K\frac{(H_o + y)}{r\cos\alpha} = 1 + K\frac{(H_o + y)}{1 + y'^2}\,y'' = 0 \qquad (4)$$

Equation (4) is the differential equation of the vacuumless spillway profile for the limiting case (no reserve for positive pressure). Equation (4) admits a simplification which throws light on the phenomenon as observed by Bazin. This is accomplished by introducing dimensionless coordinates by means of which the phenomenon may be better investigated. These coordinates are:

$$\frac{x}{H_o} = \xi \quad \text{and} \quad \frac{y}{H_o} = \eta$$

We now have:

$$dx = H_o(d\xi), \quad dx^2 = H_o^2(d\xi)^2$$

and $$dy = H_o(d\eta), \quad d^2y = H_o(d^2\eta)$$

Then: $$y' = \frac{dy}{dx} = \frac{d\eta}{d\xi} = \eta'$$

$$y'' = \frac{d^2y}{dx^2} = H_o\frac{d^2\eta}{d\xi^2} = \frac{\eta''}{H_o} \quad *$$

*Translator's note: The reader, like the translator, may be more familiar with the following method of obtaining the second derivative:

$$\frac{dy}{dx} = \frac{d\eta}{d\xi}$$

hence

$$\frac{d^2 y}{dx^2} = \frac{d}{d\xi}\,\frac{d\xi}{dx}\,\frac{d\eta}{d\xi}$$

but since $$\frac{d\xi}{dx} = \frac{1}{H_o}$$

then $$\frac{d^2 y}{dx^2} = \frac{1}{H_o}\,\frac{d^2\eta}{d\xi^2} = \frac{\eta''}{H_o}$$

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and

$$1 - \frac{K\,(H_o + y)\,y''}{1 + y^2} = 1 - \frac{K\,(H_o + H_o\eta)}{1 + \eta'^2}\,\frac{\eta''}{H_o}$$

$$= 1 - \frac{K\,(1+\eta)\,\eta''}{1+\eta'^2} = 0 \qquad (5)$$

This is the final differential equation of the vacuumless spillway profile expressed in dimensionless coordinates. The coefficient K is not a true constant; for the time being it will be considered as an unknown function of η. Consequently equation (5) can be written as follows: $K < \frac{2}{n}$

$$1 - \frac{K(\eta)\,(1+\eta)\,\eta''}{1+\eta'^2} = 0$$

In this form there is small hope of integrating the equation. Provisionally K will be assumed constant. It will be shown later that this assumption does not invalidate the final results, for by giving K various values, a whole family of curves will be found which will be intersected by the profile of the spillway.

This first integration of equation (5) is effected in the following manner:

$$\eta' = \frac{d\eta}{d\xi} = \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{d\eta}{d\alpha}\,\frac{d\alpha}{d\xi}$$

$$\eta'' = \frac{d^2\eta}{d\xi^2} = \frac{d}{d\xi}(\tan\alpha) = \frac{1}{\cos^2\alpha}\,\frac{d\alpha}{d\xi} = \frac{\tan\alpha}{\cos^2\alpha}\,\frac{d\alpha}{d\eta}$$

Hence:

$$1 - \frac{K(1+\eta)\eta''}{1+\eta'^2} = 1 - \frac{K(1+\eta)\tan\alpha\,\frac{d\alpha}{d\eta}}{(1+\tan^2\alpha)\cos^2\alpha}$$

$$= 1 - K(1+\eta)\tan\alpha\,\frac{d\alpha}{d\eta} = 0$$

But

$$\tan\alpha\,d\alpha = -d(\log\cos\alpha) = \frac{d\eta}{K(1+\eta)} = \frac{1}{K}\,d\log(1+\eta)$$

or $$d\log\left[(1+\eta)\cos\alpha^K\right] = 0 = \log 1$$

and the integral is:

$$\log\left[(1+\eta)\cos^2\alpha^K\right] = C_1$$

But when $\alpha = 0$; $\eta = 0$; therefore

$$\log 1 = C_1 = 0$$

and the integral becomes

$$\log\left[(1+\eta)\cos\alpha^{K}\right] = 0 = \log 1$$

or $$(1+\eta)\cos\alpha^{K} = 1$$

which may be written;

$$(1+\eta) = \cos\alpha^{-K} = (1+\eta'^{2})^{K/2}$$

Hence:

$$1+\eta'^{2} = (1+\eta)^{2/K}$$

and $$\eta' = \frac{d\eta}{d\xi} = \sqrt{(1+\eta)^{2/K} - 1}$$

$$= \sqrt{(1+\eta)^{n} - 1} \qquad (6)$$

where; $n = 2/K$

For two values of n the integration is performed directly. These cases are:

(a) K = 2 or n = 1

which reduces the equation to;

$$\frac{d\eta}{d\xi} = \sqrt{1+\eta-1} = \sqrt{\eta} = \eta^{1/2}$$

or $$d\xi = \frac{d\eta}{\eta^{1/2}}$$

the integral of which is

$$\xi = 2\eta^{1/2} + C_2$$

But when $\eta = 0;\ \xi = 0;$ and $C_2 = 0$

Therefore;

$$\eta = \frac{\xi^2}{4} \qquad \text{(Parabola)} \qquad (7)$$

(b) K = 1 or n = 2; then

$$\frac{d\eta}{d\xi} = \eta' = \sqrt{(1+\eta)^2 - 1} = \sqrt{1 + 2\eta + \eta^2 - 1}$$

$\sqrt{2\eta + \eta^2}$ and

$$\frac{d\eta}{\sqrt{2\eta+\eta^2}} = d\left[\log(1+\eta) + \sqrt{(1+\eta)^2 - 1}\right] = d\xi$$

and

$$\xi = \log\left[(1+\eta) + \sqrt{(1+\eta)^2 - 1}\right] + C_3$$

But when

$$\eta = 0;\ \xi = 0;\ \text{and}\ C_3 = 0$$

Finally:

$$\xi = \log\left[(1+\eta) + \sqrt{(1+\eta)^2 - 1}\right] \tag{8}$$

If we wish to consider the path of a solid particle, we proceed in the following manner: The horizontal acceleration is;

$$j_x = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv_x}{dt} = 0$$

and the initial acceleration in the y-direction is that of gravity or;

$$j_y = \ddot{y} = \frac{d^2y}{dt^2} = \frac{dv_y}{dt} = g$$

The initial velocity is:

$$v_{ox} = \sqrt{2gH_o}\ ;\ \text{and } v_{oy} = 0$$

The initial coordinates are:

$$x_o = y_o = 0$$

Then $$\frac{dv_x}{dt} = 0\ ;\ v_x = \text{const.} = \sqrt{2gH_o}$$

also: $$\frac{dv_y}{dt} = g\ ;\ v_y = gt + C_2$$

but when

$$t = 0\ ;\ v_{oy} = 0\ ;\ \text{and hence } C_2 = 0$$

Then:

$$v_y = gt\ ;\ \frac{dy}{dt} = v_y = gt$$

and $$y = \frac{gt^2}{2} + C_3$$

but when $t = o$; $y = y_o$ $=0$?; therefore $C_3 = o$,

and

$$y = \frac{gt^2}{2}$$

Likewise

$$v_x = \frac{dx}{dt} = v_{ox} = \sqrt{2gH_o}$$

Then

$$x = \sqrt{2gH_o}\,t + C_4$$

But when

$$t = 0 ;\ x = x_o = 0 ;\ \text{and}\ C_4 = 0$$

Therefore: $$X = \sqrt{2gH_o}\,t$$

or $$X^2 = 2gH_o t^2$$

and $$t^2 = \frac{x^2}{2gH_o} = \frac{2y}{g}$$

Eliminating t:

$$\frac{X^2}{H_o} = 4y ;\ \frac{X^2}{H_o^2} = \xi^2 = \frac{4y}{H_o} = 4\eta$$

and therefore

$$\eta = \frac{\xi^2}{4}$$

Therefore equation (7) indicates that the profile of the spillway with no negative pressures is a parabola which is true providing the water particles in the nappe move as a system of free solid particles in a vacuum. But since this is not the case, it indicates that the parabola, which many engineers use, has too much reserve against the formation of a vacuum, and such a profile is therefore uneconomical. Thus the parabola may be conceived as the outer limit of all vacuumless profiles.

The observed data of Bazin on which investigators usually base their design of vacuumless spillway profiles do not fit either equation $\overset{n=1}{(7)}$ or equation $\overset{n=2}{(8)}$. The Bazin curve is steeper than these curves, which indicates that differential equation (5) must be integrated with $n > 2$. This operation cannot be accomplished directly. We therefore expand the function

$$\left[(1+\eta)^{m} - 1\right]^{1/2}$$

as an infinite series, thus:

$$\eta' = \frac{d\eta}{d\xi} = \left[(1+\eta)^{m} - 1\right]^{1/2}$$

$$= \left[1 + \frac{m\eta}{1} + \frac{m(m-1)}{1.2}\eta^2 + \cdots - 1\right]^{1/2}$$

$$= \left[m\eta + \frac{m(m-1)}{1.2}\eta^2 + \frac{m(m-1)(m-2)}{1.2.3}\eta^3 \cdots\right]^{1/2}$$

$$=(m\eta)^{1/2}\left[1+\frac{m-1}{2}\eta+\frac{(m-1)(m-2)}{2\cdot 3}\eta^2+\cdots\right]^{1/2}$$

This series converges rapidly for the upper portion of the trajectory investigated by Bazin, for an inspection of Bazin's data shows that the maximum $\eta < 1$ (see table 1, Bazin's data) and, furthermore, the coefficient of the m-th term of the series is:

$$a_m=\frac{(m-1)(m-2)\cdots\cdots(m-m)}{1\cdot 2\cdot 3\cdots\cdots m(m+1)}$$

$$=\frac{\left(\frac{m}{1}-1\right)\left(\frac{m}{2}-1\right)\cdots\cdots\left(\frac{m}{m}-1\right)}{m+1}$$

and in the limit, $\lim(a_m)_{m\to\infty}=0$

The terms of the series beginning with the third, which is positive, alternate in sign, so that the omission of the terms beginning with:

$$\frac{(m-1)(m-2)}{6}\eta^2$$

will not introduce a large error when only the sum of the first two terms, namely:

$$m\eta\left(1+\frac{m-1}{2}\eta\right)$$

is taken as the approximate value of the function, $(1+\eta)^n - 1$.

With this approximate value the differential equation becomes:

$$\sqrt{n}\, d\xi = \frac{d\eta}{\sqrt{\eta + \frac{n-1}{2}\eta^2}}$$

$$= \sqrt{\frac{2}{n-1}}\, d\log\left[\tfrac{1}{2} + \frac{n-1}{2}\eta + \sqrt{\frac{n-1}{2}}\sqrt{\eta + \frac{n-1}{2}\eta^2}\right.$$

of which the integral is;

$$\sqrt{\frac{2}{n-1}}\log\left[\tfrac{1}{2} + \frac{n-1}{2}\eta + \sqrt{\frac{n-1}{2}}\sqrt{\eta + \frac{n-1}{2}\eta^2}\right] = \sqrt{n}\,\xi - C'$$

Since $(\eta)_{\xi=0} = 0$, the constant of integration, is determined from:

$$\sqrt{\frac{2}{n-1}}\log \tfrac{1}{2} + C' = 0$$

or

$$C' = -\sqrt{\frac{2}{n-1}}\log \tfrac{1}{2}$$

and

$$\sqrt{\frac{2}{n-1}}\log\left[1 + (n-1)\eta + \sqrt{n-1}\sqrt{2\eta + (n-1)\eta^2}\right]$$

$$= \sqrt{n}\,\xi \quad ; \text{or}$$

$$\xi = \sqrt{\frac{2}{n(n-1)}}\log\left[1 + (n-1)\eta + \sqrt{n-1}\sqrt{2\eta + (n-1)\eta^2}\right. \quad (9)$$

It can easily be shown that equations (7) and (8) represent special cases of this general equation. Thus, for $n = 2$, equation (9) becomes:

$$\left(\xi\right)_{n=2} = \log\left[(1+\eta) + \sqrt{(1+\eta^2) - 1}\right] \tag{8}$$

For $n = 1$, $(K = 2)$, we have the indeterminate value:

$$\xi = \sqrt{\frac{2}{0}} \log 1 = \frac{0}{0}$$

The true value of this expression is:

$$\lim \left(\xi\right)_{n \to 1} = \sqrt{2}\, \frac{\frac{d}{dn} \log\left[1 + (n-1)\eta + \sqrt{n-1}\sqrt{2\eta + (n-1)\eta^2}\right]}{\frac{d}{dn}\sqrt{n(n-1)}}$$

$$= \frac{\sqrt{2}\left[\eta + \frac{1}{2}(n-1)^{-1/2}\sqrt{2\eta + (n-1)\eta^2} + \frac{1}{2}\sqrt{n-1}\left(\eta + (n-1)\eta^2\right)^{-1/2}\eta^2\right]}{\left[1 + (n-1)\eta + \sqrt{n-1}\sqrt{2\eta + (n-1)\eta^2}\right]\frac{1}{2}\left[n(n-1)\right]^{-1/2}(2n-1)}$$

from which it follows that:

$$\lim \left(\xi\right)_{n \to 1} = \sqrt{2}\ \tfrac{1}{2}\sqrt{2\eta}. \text{ Thus } \frac{\xi^2}{4} = \eta \tag{7}$$

Calculation of the Coordinates of the Vacuumless Dam Profile

The use of formula (9) for calculating the coordinates of the curve for comparison with Bazin's data is tedious. An essential simplification is accomplished by introducing an infinite converging series for the function ξ according to the well-known expansion:

$$\log \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \quad \text{for } -1<x<1$$

For small values of x, this series converges rapidly. By putting

$$(n-1)\eta + \sqrt{n-1}\,\sqrt{2\eta + (n-1)\eta^2} = A$$

we have:

$$\frac{\xi}{\sqrt{\dfrac{2}{n(n+1)}}} = \log(1+A)$$

Now place:

$$1+A = \frac{1+x}{1-x}$$

then $$1+x = 1+A-x-xA$$

$$x(2+A) = A\;; \quad x = \frac{A}{2+A}$$

and $$\xi = 2\sqrt{\frac{2}{n(n+1)}}\left(1+\frac{x^3}{3}\right)x \qquad \text{(approximately)} \quad (9)$$

providing the series converges sufficiently rapidly so that the first two terms can be used in place of the entire series. Before making use of Bazin's table for calculating the coordinates, it is necessary to observe the following. The coordinate axes used by the author do not correspond to those of Bazin. Bear in mind that Bazin's dimensionless coordinates indicated by the subscript, B, and in addition to, his coordinate axes are different from those used by the author who employs the head, H, on the crest of the spillway rather than Bazin's h, measured at the upstream edge of the crest (see sketch in table 1). Only by converting both sets of coordinates to the same set of axes may they be compared.

The purpose of such a calculation is to find n as a function of η ; or in mathematical terms, $n = n(\eta)$. It is reasonable to assume that the law for n starts at that section where the individual stream lines deviate appreciably from a parallel state. Therefore n has been calculated for the four points in Bazin's table previous to the last point which has been excluded because apparently n is in error in the table as given;* the original source (Annales des Ponts et Chaussees, (6) 19 (1890) page 56) was not at the disposal of the author. The last column in table I gives the calculated values of n.

After the values of n have been found the problem of the author is to extend the n-law outside of that range of the nappe studied by Bazin, in consequence of which a hypothesis becomes necessary. Here lies the crux of the problem.

If the four computed values of n are plotted as a function of η , an approximate straight-line relation is obtained. However,

*Translator's note: An examination of the translation by Arthur Marichal and John C. Trautwine, Jr., of Bazin's paper shows that for $\xi_B = 0.70$, $\eta_B = -0.039$ rather than $+0.009$ as appearing in table 1 of this paper. See: Marichal, A. and Trautwine, J.C.Jr.: Recent Experiments on the Flow of Water over Weirs; Proceedings Engineer's Club of Philadelphia; Vol. 9, 1892, p. 235.

the n-law cannot be thus represented in this case, since at the lower limit of n, (n = 1), is finite, which is not possible. This can be the case only when $\eta \to \infty$. This suggests that the n-law is hyperbolic and can be expressed by:

$$(n - n_o) = \frac{a}{\eta^4} + \frac{b}{\eta^3} + \frac{c}{\eta^2} + \frac{d}{\eta} \qquad (10)$$

Observe that this series converges to n_o when $(n)_\eta \to \infty$. We know that the limiting value of n_o is 1 (free nappe with horizontal initial velocity $V_{ox} = \sqrt{2g\,H_o}$). Actually n cannot attain this value, since some frictional resistance is present. The derived equation is valid only in a certain region, for as the expression

$$\Pi = h\left(\beta \cos\alpha - \frac{\mu v^2}{r g}\right)$$

(piezometric head) shows, as h decreases, $\beta \cos\alpha$ and $\frac{\mu v^2}{r g}$ also decrease, the nappe disintegrates and moves as a system of free particles. It appears that a study of the phenomenon at spillways can yield information. Such studies have been made by Dr. Ehrenberger, the director of the State Research Station.

The use of equation (10) is very detailed and the results are not worth the trouble, since, as already stated, the n-values vary approximately as a straight line.

It is simpler to conceive of the n-line as a tangent to the hyperbola at the end point of the Bazin range (ξ = 0.450, η = 0.1115; see table 1).

We shall assume $n_0 = 1$, which is equivalent to a reserve against a negative pressure along the lower part of the spillway. With these two conditions the calculation can be carried out by means of the equation:

$$m - 1 = \frac{a}{\eta} + \frac{b}{\eta^2}$$

To find the values of a and b, we have the following conditions:

$$(1) \qquad (m-1)_{\eta=2.12} = 1.12 = \left(\frac{a}{\eta} + \frac{b}{\eta^2}\right)_{\eta=0.1115} = \frac{a}{0.1115} + \frac{b}{0.1115^2}$$

$$(2) \qquad \left(\frac{dm}{d\eta}\right)_{\eta=0.1115} = \frac{d}{d\eta}\left(\frac{a}{\eta} + \frac{b}{\eta^2}\right)_{\eta=0.1115} = -\left(\frac{a}{\eta^2} + \frac{2b}{\eta^3}\right)_{\eta=0.1115}$$

$$= \frac{(m)_{\eta=0.1115} - (m)_{\eta=0.0462}}{0.1115 - 0.0462}$$

$$= \frac{-2.32 + 2.10}{0.1115 - 0.0462} = -\frac{0.22}{0.0653} = -\frac{2200}{653}$$

The further calculation will be dispensed with here and the final result stated, thus:

$$(m-1) = \frac{0.312}{\eta} - \frac{0.00965}{\eta^2} \; * \qquad (11)$$

*Translator's note: Using the proper values appearing in table 1, equation (11) is found to be:

$$(m-1) = \frac{0.202}{\eta} - \frac{0.00848}{\eta^2} \qquad (11^1)$$

We are now ready to discuss the reserve against negative pressures. It is interesting to learn just what the analytic factor is. To do this we shall derive the equation of the reserve-curve indicated by subscript 1 and consider η as the independent variable. Thus we may write:

$$1 - \frac{k(1+\eta_1)\eta_1''}{1+\eta_1'^2} > 0; \quad \frac{\eta_1''}{1+\eta_1'^2} < \frac{1}{k(1+\eta_1)}$$

or $$\frac{\eta_1''}{1+\eta_1'^2} = \frac{a}{k(1+\eta_1)} \quad \text{where } a < 1$$

Then: $$1 - \frac{k(1+\eta_1)\eta_1''}{a(1+\eta_1'^2)} = 0$$

The first integral of this equation is:

$$\eta_1' = \frac{d\eta_1}{d\xi} = \sqrt{(1+\eta_1)^{n_1} - 1} \quad \text{where } n_1 = \frac{2a}{k} < n$$

Therefore, any curve determined from differential equation (6) with $n_1 < n$ is a reserve-curve; the limiting curve rises steeper than the reserve-curve.

After the n-law has been found, the coordinates are calculated as far as $\eta \leqq 0.80$ by equation (9). For $\eta \geqq 0.80$ a new formula must be employed. To do this we write equation (6) in the form:

$$\eta' = \frac{d\eta}{d\xi} = \sqrt{(1+\eta)^{n} - 1}$$

$$\frac{d\eta}{\sqrt{(1+\eta)^m - 1}} = d\eta\left[(1+\eta)^m - 1\right]^{-1/2} = d\xi$$

Expanding by an infinite series we have:

$$\left[(1+\eta)^m - 1\right]^{-1/2} = (1+\eta)^{-m/2}\left[1 - \frac{1}{(1+\eta)^m}\right]^{-1/2}$$

$$= (1+\eta)^{-m/2}\left[1 + \frac{1}{2}\frac{1}{(1+\eta)^m} + \frac{1.3}{2.2.2(1+\eta)^{2m}}\right.$$

$$\left. + \frac{1.3.5}{2.2.2.1.2.3(1+\eta)^{3m}} + \cdots \frac{1.3.5\ldots(2m-1)}{2^m\ m!\ (1+\eta)^{m\cdot m}}\right]$$

This series is convergent because the coefficient:

$$\frac{1.3.5\ldots(2m-3)(2m-1)}{2^m\ 1.2.3\ldots(m-1)m} < 1$$

in which

$$\lim\left(\frac{2m-1}{2m}\right)_{m\to\infty} = 1$$

Then:

$$\int_{0.50}^{\eta}\frac{d\eta}{\sqrt{(1+\eta)^m - 1}} = \int_{0.50}^{\eta}(1+\eta)^{-m/2}\,d\eta$$

$$+ \frac{1}{2}\int_{0.50}^{\eta}(1+\eta)^{-3m/2}\,d\eta + \cdots$$

$$+ \frac{1.3.5\ldots(2m-3)(2m-1)}{2^m\, m!} \int_{0.50}^{\eta} (1+\eta)^{-\left(\frac{1+2m}{2}\right)n}\, d\eta$$

$$= \frac{1}{1-n/2} \Big| (1+\eta)^{1-n/2} \Big|_{0.50}^{\eta} + \frac{1}{2(1-3n/2)} \Big| (1+\eta)^{1-3n/2} \Big|_{0.50}^{\eta}$$

$$+ \ldots \frac{1.3.5\ldots(2m-1)}{2^m\, m!\left(1-\frac{2m+1}{2}n\right)} \Big| (1+\eta)^{1-\frac{2m+1}{2}n} \Big|_{0.50}^{\eta}$$

$$= \frac{2}{2-n}\left[(1+\eta)^{\frac{2-n}{2}} - 1.5^{\frac{2-n}{2}}\right] + \frac{2}{2(2-3n)}\left[(1+\eta)^{\frac{2-3n}{2}} - (1.5)^{\frac{2-3n}{2}}\right]$$

$$+ \ldots + \frac{1.3.5\ldots(2m-1)\,2}{2^m\, 1.2.3\ldots m\,[2-(2m+1)n]}$$

$$\left[(1+\eta)^{\frac{2-(2m+1)n}{2}} - 1.5^{\frac{2-(2m+1)n}{2}}\right]$$

$$= (\xi)_{\eta} - (\xi)_{\eta=0.50} \qquad (12)$$

The coordinates for $\eta > 0.50$ are computed from formula (12).

It is now appropriate to say a few words about how the difficulties are overcome when we calculate η by assuming n to be constant, instead of by using equation (11). This is profitable only insofar as one point on a curve computed with n = constant, has an η belonging to the vacuumless profile which satisfies equation

(11); all of the remaining points on the curve are not points on the profile curve. Following this procedure we obtain a whole family of curves which intersect the profile curve.

The coordinates calculated according to the author's equations are given in table 2. In this table coordinates with and without subscripts are encountered; the coordinates without a subscript are calculated for H; those with a subscript, for h, as indicated in the sketch accompanying the table. The values for h are necessary in order to compare the coordinates with those of other authors.

The coordinates for the parabolic equation are also included in the table in order to show directly and indirectly that such a curve is uneconomical; directly, by increasing the dimensions of the dam - and indirectly, by the unfavorable influence on the discharge coefficient, μ.

The profiles advocated by various writers are given in figure 5, among them the author's curve, of which it may be said that it is truly a safe profile, since at first it agrees with Maxim's profile and farther along it gradually deviates from the limiting curve in such a way as to increase the reserve against the formation of negative pressure.

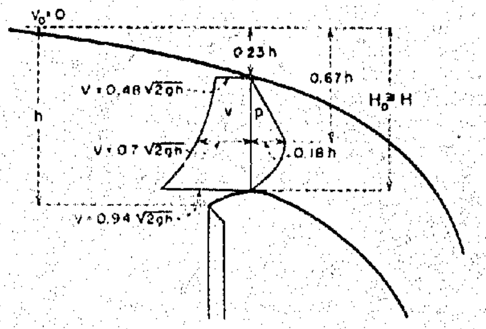
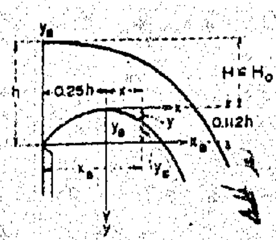


FIGURE 1 - DISTRIBUTION OF VELOCITY AND PRESSURE WITHIN A FREE NAPPE.

TABLE I
BAZIN'S COORDINATES FOR THE UNDER SURFACE OF THE NAPPE AND COMPUTED VALUES OF n.

HEIGHT OF CREST OF WEIR = 1.13 m HEAD ON WEIR, h = 0.15 - 0.45 m				
FOR BAZIN'S AXES		FOR VITOL'S AXES		n
ξ_B	η_B	ξ	η	
0.05	0.059			
0.10	0.085			
0.15	0.101			
0.20	0.109			
0.25	0.112	0.000		
0.30	0.111	0.056	0.00113	
0.35	0.106	0.113	0.00675	
0.40	0.097	0.169	0.0169	
0.45	0.085	0.225	0.0304	
0.50	0.071	0.282	0.0462	2.32
0.55	0.054	0.338	0.0653	2.24
0.60	0.035	0.395	0.0867	2.20
0.65	0.013	0.450	0.1115	2.10
0.70	0.009	0.506	0.132	



$x_B - x = 0.25\,h$

$y_B + y = 0.112\,h$

From which it follows:

$$\frac{x_B}{h} = \frac{x}{H_0}\frac{H_0}{h} = \xi_B - \xi\,0.888 = 0.25;$$

$$\text{or } \xi = \frac{\xi_B - 0.25}{0.888}$$

Similarly:

$$\eta = \frac{0.112 - \eta_B}{0.888}$$

FORMULAS FOR TRANSFORMATION OF AXES.

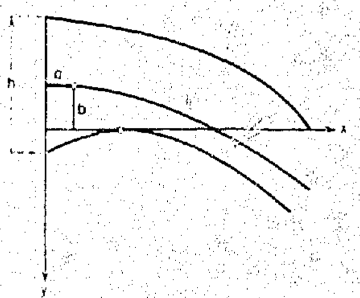
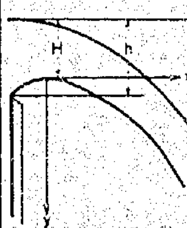


FIGURE 2 - THE CREAGER PARABOLA.

TABLE II
COMPUTED COORDINATES

ACCORDING TO FORMULAS (9) AND (12)				ACCORDING TO FORMULA (7) (PARABOLA)			
$\xi = \frac{x}{H_0}$	$\eta = \frac{y}{H_0}$	$\xi_n = 0.888\xi$	$\eta_n = 0.888\eta$	$\xi = \frac{x}{H_0}$	$\eta = \frac{y}{H_0}$	$\xi_n = 0.888\xi$	$\eta_n = 0.888\eta$
0.0563	0.00123	0.050	0.001	0.200	0.01	0.1776	0.0888
0.0126	0.00675	0.100	0.006	0.400	0.04	0.3552	0.0355
0.169	0.0169	0.150	0.015	0.600	0.09	0.5328	0.0799
0.225	0.0304	0.200	0.027	0.800	0.16	0.7104	0.1421
0.282	0.0462	0.250	0.041	1.000	0.25	0.8880	0.2220
0.338	0.0653	0.300	0.058	1.200	0.36	0.0656	0.3200
0.395	0.0867	0.350	0.077	1.400	0.49	1.2432	0.4350
0.450	0.115	0.400	0.099	1.600	0.64	1.4208	0.5683
1.175	0.500	1.043	0.444	1.800	0.81	1.5984	0.7200
1.770	1.00	1.572	0.888	2.000	1.00	1.7760	0.8880
2.617	2.00	2.324	1.776	2.828	2.00	2.5120	1.776
3.240	3.00	2.877	2.664	3.464	3.00	3.0760	2.664
3.721	4.00	3.320	3.552	4.000	4.00	3.5520	3.552
4.277	5.00	3.800	4.440	4.472	5.00	3.9600	4.440



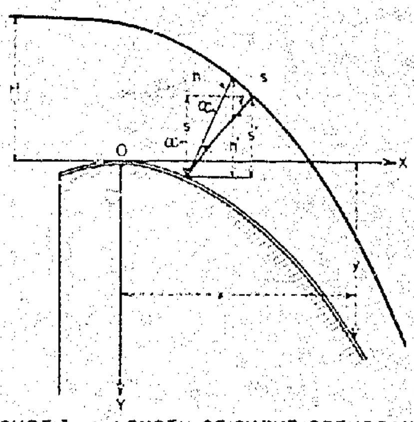


FIGURE 3 - s = LENGTH OF CURVE ORTHOGANAL TO STREAM LINES; h = THE THICKNESS OF THE NAPPE NORMAL TO THE SPILLWAY PROFILE; s' AND h' = THEIR VERTICAL PROJECTIONS.

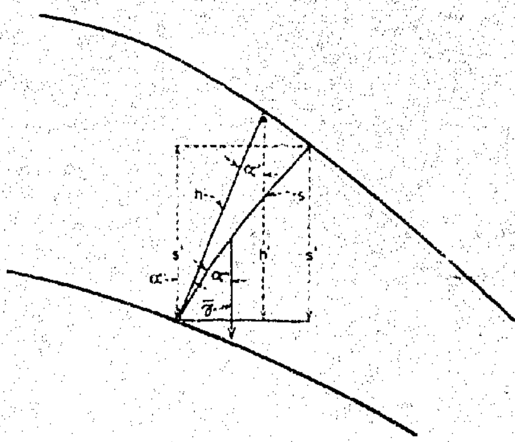


FIGURE 4 - GEOMETRICAL EXPLANATION OF SYMBOLS.

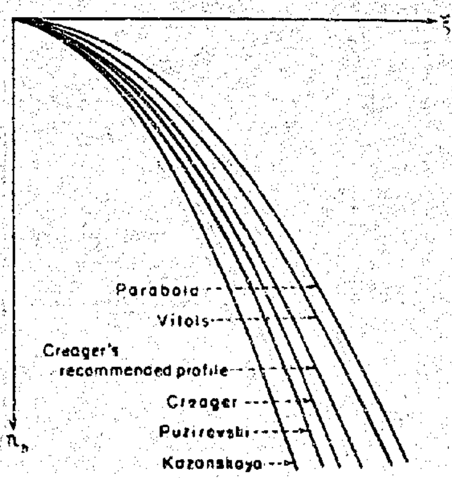


FIGURE 5 - VACUUMLESS PROFILES ACCORDING TO VARIOUS AUTHORS.