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CONTRIBUTION TO THE THEORY OF THE FLOW IN OPEN CHANNELS AND PIPES

- - -

A Translation From German

by

EDWARD F. WILSEY

- - -

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A Translation of

BEITRAG ZUR THEORIE DER BEWEGUNG DES WASSERS IN OFFENEN KANÄLEN UND ROHRLEITUNGEN

By

FELIX ZILLER

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CONTRIBUTION TO THE THEORY OF THE FLOW IN OPEN CHANNELS AND PIPES

Synopsis

The impulse principle is derived from the fundamental equations of dynamics. An equation for the slope of the water surface is derived and discussed for several special cases, but it is only valid for flat slopes. However, it can be extended to greater slopes and can be applied in this new form to the flow in pipes. Finally, several examples are discussed.

The Impulse Principle

Leonhard Euler developed one of the most important principles in the theory of the flow of water. He applied the basic equations of dynamics to an infinitesimal volume of a fluid whose position is defined by the vector, $\bar{r}$, or by the corresponding xyz-coordinates, and he obtained the equations known today in hydrodynamics as the Euler equations.

Euler's theory, although only applicable to the flow of a frictionless fluid, can easily be transferred to flow possessing frictional resistance. A law of dynamics, when expressed vectorially, shows that,

$$\frac{d}{dt}(m\,\bar{v}) = \bar{G} + \bar{K} + \bar{F} \qquad (1)$$

in which $\bar{G}$ = the force of gravity vector.
$\bar{K}$ = the pressure force vector.
$\bar{F}$ = the frictional force vector.
t = the time.
m = the mass.
$\bar{v}$ = the velocity vector.

This relation is evidently valid for an infinitely small element of volume, $d\tau$, and then its form is as follows:

$$\frac{d}{dt}\left(\frac{\gamma}{g}\bar{v}\,d\tau\right) = \gamma\,d\tau + d\bar{P} + d\bar{F}$$

in which γ = the specific weight of the fluid.
g = the acceleration of gravity.

If we now integrate over any suitably chosen region, B, within the fluid, we obtain the impulse principle; thus

$$\frac{\gamma}{g}\int^{B}\frac{d}{dt}\left(d\tau\,\bar{v}\right) = \int^{B}\gamma\,d\tau + \oint^{B} d\bar{P} + \bar{F} \qquad (2)$$

The two integrals on the right-hand side of the equation can be evaluated directly as external forces. $\bar{F}$ must be considered in the light of physical data already obtained on fluid resistance. The left-hand side of the equation can be further transformed. Thus

$$d\left(d\tau\,\bar{v}\right) = \frac{\partial(\bar{v}\,d\tau)}{\partial t}\,dt + \overline{\text{grad}}\,(\bar{v}\,d\tau)\cdot dr$$

$$\frac{dr}{dt} = \bar{v}$$

$$\frac{d(d\tau\,\bar{v})}{dt} = \frac{\partial(d\tau\,\bar{v})}{\partial t} + \overline{\text{grad}}\,(d\tau\,\bar{v})\cdot\bar{v}$$

Consequently, if the first term is differentiated before the integration is performed, we may write

$$\int^{B}\frac{d}{dt}\left(d\tau\,\bar{v}\right) = \frac{\partial}{\partial t}\int^{B}\bar{v}\,dt + \int^{B}\left(\overline{\text{grad}}\,\bar{v}\right)\bar{v}\,d\tau$$

The total impulse, I, in the region B of the fluid is,

$$\frac{\gamma}{g}\int^{B} \bar{v}\, dt$$

However, it can be proved* that,

$$\int^{B} (\overline{grad}\ \bar{v})\,\bar{v}\, d\tau = \oint^{B} \bar{v}\,\bar{v}\cdot d\sigma$$

Since by definition,

$$\bar{v}\, d\sigma = dQ$$

where Q is the discharge per second, the impulse principle assumes the following form

$$\frac{\partial I}{\partial t} + \frac{\gamma}{g}\oint^{B} \bar{v}\, dQ = \bar{G} + \oint^{B} d\bar{P} + \bar{F} \qquad (3a)$$

*Proof of the equation,

$$\int^{B} (\overline{grad}\ \bar{v})\,\bar{v}\, d\tau = \oint^{B} \bar{v}\,\bar{v}\cdot d\sigma$$

$\bar{a}$ and $\bar{b}$ are any two vectors and e_i is a unit vector. Cartesian coordinates are used throughout.

Put

$$\bar{a} = \sum_i e_i\, a_i \quad ; \quad \bar{b} = \sum_k e_k\, b_k$$

Then from the tensor $\bar{a}\bar{b}$, we have

$$\bar{a}\,\bar{b} = \sum_{ik} e_i\, e_k\, a_i\, b_k$$

We can now evaluate

$$\nabla \cdot (\bar{a}\,\bar{b})$$

in which

$$\nabla = \sum e_j \frac{\partial}{\partial x_j}$$

and obtain

$$\nabla \cdot (\bar{a}\,\bar{b}) = \sum_k \left(e_k b_k \sum_i \frac{\partial a_i}{\partial x_i} \right) + \sum_i \left(a_i \sum_k \frac{\partial b_k}{\partial x_i} e_k \right)$$

$$= \bar{b}\, \overline{\mathrm{div}}\, \bar{a} + \bar{a} \cdot (\overline{\mathrm{grad}}\, \bar{b})$$

By analogy

$$(\bar{a}\,\bar{b}) \cdot \nabla = \bar{a}\, \overline{\mathrm{div}}\, \bar{b} + (\overline{\mathrm{grad}}\, \bar{a}) \cdot \bar{b}$$

Hence

$$(\bar{v}\,\bar{v}) \cdot \nabla = \nabla \cdot (\bar{v}\,\bar{v}) = \bar{v}\, \overline{\mathrm{div}}\, \bar{v} + (\overline{\mathrm{grad}}\, \bar{v}) \cdot \bar{v}$$

By excluding compressible fluids and flow from sources, the equation of continuity is written,

$$\overline{\mathrm{div}}\, \bar{v} = 0$$

and

$$(\overline{\mathrm{grad}}\, \bar{v}) \cdot \bar{v} = \nabla \cdot \bar{v}\,\bar{v}$$

Therefore Gauss' equation becomes

$$\int^{B} (\overline{\text{grad}}\ \bar{v})\bar{v}\, d\tau = \oint^{B} \bar{v}\bar{v} \cdot d\sigma$$

Put

$$\frac{\gamma}{g}\bar{v}\, dQ = dI'$$

Then

$$\frac{\partial I}{\partial t} + \oint^{B} dI' = \bar{G} + \int^{B} d\bar{P} + \bar{F} \qquad (3b)$$

The first integral is called the impulse transport.

Equation (3b) is the most general form of the impulse principle for the sourceless motion of an incompressible fluid. Special types of flow can be easily defined on the basis of this equation: if $\frac{dI}{dt} = 0$, the flow is steady. Flow over a weir or under a sluice gate, and motion of water in reservoirs created by dams, can usually be considered steady.

On the other hand, if $\oint^{B} dI' = 0$, we have uniform motion. This is the case presented by the pulsations of the mass of water in a penstock (water hammer problem). With steady uniform flow as in a canal with constant cross section and slope, the external forces must be in equilibrium.

The application of the impulse principle to a problem in hydraulics of the most general type proceeds as follows:

1. Establish the boundary of the fixed region B, called the "control area."

2. Calculate the external forces acting on the fluid within the region, B, in other words, compute the force of gravity, $\bar{G}$ the pressure force, $\oint^{B} d\bar{P}$, and the frictional force, $\bar{F}$.

3. Calculate the impulse as a function of the time and form $\frac{\partial I}{\partial t}$.

4. Determine the impulse transport $\oint^{B} dI'$.

It must not be forgotten that the use of the impulse principle presupposes that either the pressure or the velocity vector is known as a function of the position. As a rule, the pressure is calculated knowing the velocity distribution. It can be determined by means of the equation of continuity $\overline{\text{div}}\ \bar{v} = 0$ as long as no reasonable assumption is possible but this often introduces considerable mathematical difficulties.

Application of the Impulse Principle to an Open Channel

Given the curve of the bottom, the dimensions of the cross section, and the discharge, Q; to find the profile of the water surface. In order to solve this problem, we place two vertical sections perpendicular to the axis of the canal and Δx apart. Their areas are A_n and A_{n+1}, respectively.

1. The volume of water contained between these two planes is the region of integration, B.

2. External forces.

Force of gravity.--The force of gravity is given at once by

$$\bar{G} = \gamma \int_0^{\Delta x'} \int_0^{s} p(\xi)\, dx\, d\xi$$

$$= \gamma (A_n + A_{n+1}) \frac{\Delta x}{2}$$

Pressure force.--The force due to the pressure acting on A_n is,

$$P_n = \gamma \int^{A_n} \eta \, dA_n = \gamma \int_0^{S_n} \frac{[\rho(\xi)]_n^2}{2} d\xi$$

and on A_{n+1} is,

$$P_{n+1} = \gamma \int^{A_{n+1}} \eta \, dA_{n+1} = \gamma \int_0^{S_{n+1}} \frac{[\rho(\xi)]_{n+1}^2}{2} d\xi$$

Place

$$[\rho(\xi)]_{n+1} - [\rho(\xi)]_n = \Delta T$$

Then

$$\frac{1}{2}[\rho(\xi)]_{n+1}^2 = \frac{1}{2}\left\{[\rho(\xi)]_n^2 + 2[\rho(\xi)]_n \Delta T + (\Delta T)^2\right\}$$

Therefore

$$P_{n+1} - P_n =$$

$$\gamma \Delta T \left\{ \int_0^{S_n} [\rho(\xi)]_n d\xi + \int_0^{S_{n+1}} \left([\rho(\xi)]_n + \Delta T\right) d\xi \right\} \frac{1}{2} =$$

$$\gamma T \frac{1}{2} (A_n + A_{n+1})$$

The force of the pressure on the bottom is

$$P_s = \gamma \int_0^s [\rho(\xi)]\, dx\, d\xi \frac{1}{\cos\phi} =$$

$$\frac{\gamma \Delta x}{\cos\phi}\left\{\int_0^{s_n} [\rho(\xi)]_n\, d\xi + \int_0^{s_{n+1}} [\rho(\xi)]_{n+1}\, d\xi\right\} \frac{1}{2}$$

The vertical component of this force is

$$\gamma \int_0^{\Delta x}\int_0^s [\rho(\xi)]\, dx\, d\xi = |\bar{G}|$$

The horizontal component is

$$\tan\phi\ \gamma \int_0^{\Delta x}\int_0^s [\rho(\xi)]\, dx\, d\xi = |\bar{G}| \tan\phi$$

Frictional force.--In order to find an expression for the frictional force, we assume that the resistance law for uniform steady flow can be transferred directly to nonuniform nonsteady flow. According to the equation already found above

$$\bar{F} = \bar{G} \tan\phi$$

in which

$$\bar{G} = \frac{\gamma}{2}\left(A_n + A_{n+1}\right)\Delta x$$

and $\tan\phi$ = the slope, i. i can be computed according to the formulas published by a number of authors. Thus Hopf and Fromm[2] give

$$i = 10^{-2}\left(\frac{K'}{R}\right)^{0.314} \frac{V^2}{2g} \frac{1}{R}$$

(R = the hydraulic radius computed as the ratio of area of flow to the wetted perimeter). Or

$$i = \frac{10^{-2}(K')^{0.314}}{2} \frac{V^2}{gR^{1.314}} = \frac{1}{cg} \frac{V^2}{R^{1.314}}$$

According to Strickler's formula

$$V = C' R^{2/3} i^{1/2}$$

or

$$i = \frac{V^2}{C'^2 R^{1.33}}$$

Thus F is computed from

$$F = \frac{1}{2}(A_n + A_{n+1})\gamma \frac{V^2}{cgR^{1.314}} \qquad (6)$$

3. <u>Impulse I and</u> $\frac{\partial I}{\partial t}$

$$\frac{\partial I}{\partial t} = \frac{\gamma}{g} \frac{\partial}{\partial t} \int_0^{\Delta x} \int_0^{A} \bar{v}\, dA\, dx = \frac{\gamma}{g} \frac{\partial}{\partial t} \int_0^{\Delta x} Q\, dx \qquad (7)$$

4. <u>Impulse transport.</u> The impulse transport is uniquely dependent on the areas A_n and A_{n+1}. If we assume that the velocity distribution at a cross section is known, then at section A_n we have

$$I'_n = \frac{\gamma}{g} \int^{A_n} V_{xn}\, dA$$

Since it is not convenient to integrate, we introduce the average velocity

$$V = \frac{Q}{A_n}$$

Then

$$I'_n = \frac{\gamma}{g} V_n^2 A_n \alpha_n \tag{8}$$

α_n is a correction factor which takes into account the nonuniformity of the velocity distribution over the whole cross section. It is found from

$$\alpha_n = \frac{\int^A V_{xn}^2 \, dA}{A_n V_n^2}$$

Therefore we use I'_n in the following form

$$I'_n = \alpha_n \frac{\gamma}{g} V_n Q_n = \alpha_n \frac{\gamma}{g} V_n^2 A_n = \alpha \frac{\gamma}{g} \frac{Q_n^2}{A_n}$$

The impulse equation becomes

$$\frac{\partial}{\partial t}\int_0^{\Delta x} Q \, dx \frac{\gamma}{g} + \frac{\gamma}{g}\left(V_{n+1} Q_{n+1} \alpha_{n+1} - V_n Q_n \alpha_n\right) =$$

$$-\Delta x \gamma \left(A_n + A_{n+1}\right) \frac{\tan \phi}{2} - \frac{\Delta T \gamma}{2}\left(A_n + A_{n+1}\right)$$

$$- \frac{1}{2}\left(A_n + A_{n+1}\right) \gamma \frac{V^2 \Delta x}{C g R^{1.314}}$$

But from figure 1

$$\Delta T = \Delta Z - \Delta y \; ; \; \Delta x \tan \phi = \Delta y$$

Introducing these equalities and cancelling we have

$$\frac{\partial}{\partial t}\int_0^{\Delta x} Q\,dx + Q_{n+1}\alpha_{n+1} - QV_n\alpha_n =$$

$$-\Delta Z\left(A_n + A_{n+1}\right)\frac{g}{2} - \frac{1}{2}\left(A_n + A_{n+1}\right)\frac{V^2}{CR^{1.314}}\Delta x \qquad (9)$$

Divide through by Δx and let $\Delta x \to 0$ and then neglect all terms containing infinitesimals of the second order or higher. Then

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(QV\alpha\right) + \frac{AV^2}{CR^{1.314}} = -\frac{\partial Z}{\partial x}Ag \qquad (10a)$$

Since A is a function of (z-y), we have the following relations

$$V = \frac{Q}{A}$$

$$\frac{\partial V}{\partial x} = \frac{1}{A}\frac{\partial Q}{\partial x} - \frac{Q}{A^2}\frac{\partial A}{\partial x}$$

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial (z-y)}\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial x}\right)$$

Also

$$\frac{\partial A}{\partial (z-y)} = \lim_{\Delta(z-y)\to 0}\frac{\Delta A}{\Delta (z-y)} = s$$

$$\frac{\partial V}{\partial x} = \frac{1}{A}\frac{\partial Q}{\partial x} - \frac{Q}{A^2}s\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial x}\right)$$

Therefore

$$\frac{\partial Q}{\partial t} + \frac{Q^2}{A^2}\alpha S \frac{\partial y}{\partial x} + 2\alpha \frac{Q}{A}\frac{\partial Q}{\partial x} + \frac{\partial \alpha}{\partial x}\frac{Q^2}{A} + \frac{Q^2}{A C R^{1.314}}$$

$$= - \frac{\partial z}{\partial x}\left(A g - \frac{\alpha Q^2}{A^2} S\right) \tag{10b}$$

The equation of continuity is

$$\frac{\partial Q}{\partial x} = - S \frac{\partial z}{\partial t} \tag{11}$$

Differentiate (10b) again with respect to x and equation (11) with respect to t and x. Then introduce (11) into (10b). After a single transformation, the following expression is obtained:

$$\frac{\partial^2 z}{\partial t^2} + f_1\left(\frac{\partial z}{\partial t}\right)^2 + f_2 \frac{\partial z}{\partial t} = f_3 \frac{\partial^2 z}{\partial x^2} + f_4\left(\frac{\partial z}{\partial x}\right)^2 + f_5 \frac{\partial z}{\partial x} + f_6$$

$$\text{where} \quad f_n = f(Q, z, y) \tag{12}$$

This is the general differential equation for nonuniform nonsteady flow, especially for wave motion, in a canal whose cross section varies in any way. However, a general solution is not possible.

Small Disturbances[3]

We select a particularly simple case of equation (12). Given a canal of constant cross-sectional area, A, and a flat bottom. Let Z_0 be the initial depth of flow. If this condition is disturbed at any place X, the disturbance propagates itself as a wave. We assume further that the maximum variation in depth is very small in comparison to the depth of flow. Then

$$\frac{\partial}{\partial x}(Q V \alpha)$$

and the effect of the frictional resistance can be neglected. Hence we obtain

$$\frac{\partial^2 Z}{\partial t^2} = \frac{A}{S} g \frac{\partial^2 Z}{\partial x^2} \tag{13}$$

The solution of this differential equation is

$$Z = f_1(x - \omega t) + f_2(x + \omega t)$$

which satisfies (13) when

$$\omega^2 = \frac{A}{S} g$$

The initial conditions are the position X_1; the time t_1; and the depth,

$$Z = f_1(x_1 - \omega t) + f_2(x_1 + \omega t).$$

An equal depth will occur at X_2 at some later time, t_2. Therefore

$$x_1 - \omega t_1 = x_2 - \omega t_2$$

or

$$x_1 + \omega t_1 = x_2 + \omega t_2$$

and

$$\omega = \pm \frac{x_2 - x_1}{t_2 - t_1}$$

The disturbance propagates itself with a velocity, w. If before the disturbance occurs there is a velocity, V_o, in the channel, the velocity of travel of the wave is

$$V_o + \sqrt{g \frac{A}{S}} \tag{14a}$$

or, if the channel has a rectangular cross section

$$V_o + \sqrt{gT} \tag{14b}$$

Steady Flow

With steady flow

$$\frac{\partial Q}{\partial t} = 0$$

and also according to (11)

$$\frac{\partial Q}{\partial x} = 0$$

Divising (10b) by A_g, we have

$$-\frac{\partial z}{\partial x}\left(1 - \frac{Q^2 \alpha S}{g A^3}\right) = \frac{Q^2 \alpha S}{g A^3}\frac{dy}{dx} + \frac{d\alpha}{dx}\frac{Q^2}{A^2 g} + \frac{Q^2}{A^2 g C R^{1.314}} \tag{15}$$

This equation can be used to compute backwater curves[4] when α is assumed constant. Also place

$$\frac{\partial z}{\partial x} = \frac{\Delta z}{\Delta x}$$

and obtain the following relation

$$\Delta x = \frac{-\Delta z\left(\frac{A^2}{Q^2} - \frac{\alpha S}{gA}\right)}{\frac{\alpha S}{gA}\tan\phi + \frac{1}{C g R^{1.314}}} \tag{16}$$

which is a form similar to that found by Hagen[5]. The numerical computation proceeds in the following steps:

1. Choose a suitable ΔZ
2. Calculate A, R, and S for Z_0 and then determine ΔX.
3. Find $Z_1 = Z_0 + \Delta Z$ and repeat the process.

A value of X is found for each corresponding value of Z and when plotted give the backwater curve (figure 2). Although the value of α is about 1.06, it is usually taken as unity. If the average velocity, Q/A, is small, the following relation may be introduced:

$$\frac{Q^2 S \alpha}{g A^2} = 0$$

The result is the approximate formula much used in hydraulics for computing backwater curves, or

$$-\frac{dz}{dx} = \frac{Q^2}{A^2 c g R^{1.314}}$$

Since $Z = y + d$

$$\frac{dz}{dx} = \frac{dy}{dx} + \frac{dD}{dx}$$

Then

$$\frac{dD}{dx} = -\frac{dy}{dx} - \frac{Q^2}{A^2 c g R^{1.314}}$$

But

$$\frac{dy}{dx} = S$$

Hence

$$\Delta X = \frac{\Delta D}{S - Q^2/A^2 c g R^{1.314}} \qquad (17)$$

Lateral inflow or outflow can be easily included in the case of steady flow. Then the discharge in the canal varies with X and dQ. If V' is the projection of the lateral inflow velocity of the inflow discharge, dQ, on the velocity in the canal, then

$$-\frac{dz}{dx} = \frac{V^2}{g}\left[\frac{\alpha}{V}\frac{dV}{dx} + \left(\alpha - \frac{V'}{V}\right)\frac{1}{Q}\frac{dQ}{dx} + \frac{d\alpha}{dx} + \frac{1}{CR^{1.314}}\right] \quad (18)$$

This equation was first derived by Henri Favre[6] in his "Contribution a l'Etude des Courants Liquids" (Contributions to the Study of the Flow of Liquids"), Zurich.

Shooting and Streaming Flow[7]

Neglecting the friction loss, (15) becomes

$$-\frac{\partial z}{\partial x}\left(1 - \frac{Q^2 \alpha S}{A^3 g}\right) = \frac{Q^2 \alpha S}{A^2 g}\frac{dy}{dx}$$

or

$$-\frac{\partial z}{\partial x}\left(gA^3 - Q^2 \alpha S\right) = Q^2 \alpha S \frac{dy}{dx}$$

Some general conclusions can be drawn from this equation.

1. If the slope of the water surface is zero, the bottom slope must also be zero or Q must be zero, that is, the water is at rest.

2. If the slope of the bottom is zero, $\frac{dy}{dx} = 0$,

the slope of the water surface must be zero or

$$\left(gA^3 - Q^2 \alpha S\right) = 0$$

Then

$$\frac{A}{g\alpha S} = \frac{Q^2}{A^2} = V^2$$

or

$$V = \sqrt{g \frac{A}{\alpha S}} \tag{19}$$

This formula, when $\alpha = 1$, agrees with the velocity of propagation of small disturbances in still water.

If

$$V < \sqrt{g \frac{A}{S}}$$

then

$$\left(g A^3 - Q^2 \alpha S\right) > 0$$

A disturbance moves downstream with a velocity

$$\sqrt{g \frac{A}{S}} + V$$

and upstream with a velocity

$$\sqrt{g \frac{A}{S}} - V$$

The signs of the surface slope and the bottom slope are opposite if

$$V > \sqrt{g \frac{A}{S}}$$

then

$$\left(gA^3 - Q^2 \alpha s\right) < 0$$

and a disturbance moves with a negative velocity

$$\sqrt{g\frac{A}{S}} - V$$

Therefore it does not move upstream. The surface slope, therefore, has the same sign as the bottom slope.

As proposed to Rehbock when

$$\left(gA^3 - Q^2 \alpha s\right) > 0$$

the flow is said to be streaming and when

$$\left(gA^3 - Q^2 \alpha s\right) < 0$$

it is said to be shooting.

In order to simplify this concept we shall consider in the following a canal of rectangular cross section. The transition from shooting to streaming is when

$$V = \sqrt{gD}$$

The depth at this transitional velocity is called the critical depth; it is computed from

$$d_{cr} = \sqrt[3]{\frac{Q^2 \alpha}{S^2 g}}$$

and the type of flow can be called

Streaming flow if $D > d_{cr}$
Shooting flow if $D < d_{cr}$

If a hump is placed on the previous level floor of the canal and the cross section is rectangular, three types of overflow jets are possible.

Case I. If $(gA^3 - Q^2 \alpha S)$ is always positive, that is, if $d > dcr$, streaming flow persists throughout and the slope of the water surface and the bottom slope have opposite signs.

Case II. If $(gA^3 - Q^2 \alpha S)$ is always negative, that is, if $d < dcr$, shooting flow exists throughout and the slopes of the water surface and the bottom have the same sign (a rare condition).

Case III. If $(gA^3 - Q^2 \alpha S)$ changes sign, at the crest of the hump, where $dy/dx = 0$,

$$(gA^3 - Q^2 \alpha S) = 0 \text{ and } D = dcr$$

Thus the flow changes from a streaming to a shooting condition.

Since the concept of a weir with a rounded crest is fundamentally not far removed from that of a hump in the floor, the flow pattern being as in III, an attempt can be made to develop a weir formula using the assumption that the critical depth takes place at the crest of the weir (Bundschu)[8]. Of course such a theory is only roughly approximate, since the curvature of the stream lines, which is important with this type of weir, is neglected though this is not permissible, as experimental results show.

However, the curvature of the stream lines can be neglected with some justification for very low sills, and from case III we obtain

$$Q = S\sqrt{g\, d_{cr}^3} \tag{20a}$$

Introducing the height, h, of the energy gradient above the crest of the weir, we have

$$d_{cr} = \frac{2}{3} h$$

With this

$$Q = S\,0.385\sqrt{2gh^3} \tag{20b}$$

The situation when a rectangular canal of variable width, S, and a level floor (y = o) is used, is entirely analogous.

Formula (10a) then reads

$$\frac{dz}{dx} = \frac{Q^2}{gA^3}\,\alpha\,\frac{dA}{dx}$$

$$A = zS$$

$$\frac{dA}{dx} = S\frac{dz}{dx} + z\frac{dS}{dx}$$

$$\frac{dz}{dx}\left(gA^3 - Q^2\alpha S\right) = Q^2\alpha z\frac{dS}{dx}$$

As before, three possible patterns of flow exist which are shown in figure 4.

These patterns show a complete correspondence to those in figure 3. However it is to be noticed that with type I, the depth decreases as the width decreaes; with type II, the depth increases with decreasing width. For type III, either discharge formula (20a) or (20b) is applicable, since the critical depth occurs at the narrowest section.

In most cases, however, when the cross section is narrowed, $d\alpha/dx$ and often the friction term $1/CR^{1.314}$ must be taken into consideration. For the present, particularly if $d\alpha/dx$ enters, these factors can only be considered from an experimental standpoint. Thus the above analysis is only of theoretical value.

Extension to Steep Bottom Slopes[9]

Although all of the foregoing formulas are not general, since

they were derived for channels of very flat slope, they possess, however, at least some practical significance for open channels. For steep slopes, the force of friction must be resolved into horizontal and vertical components. This means that the pressure on the bottom, represented by the pressure head h (ξ), is no longer equal to the depth of flow as given by the equation of the profile of the cross section p (ξ). Since the friction acts parallel to the direction if flow, h (ξ) is best determined in the following manner:

$$\int_0^S h(\xi)\,dx\,\frac{1}{\cos\phi}\,d\xi = \int_0^S p(\xi)\,dx\,\overline{\cos}\,\phi\,d\xi$$

and

$$h(\xi) = p(\xi)\,\overline{\cos}^2\phi \tag{22}$$

As can be easily shown, the pressure distribution along a vertical is triangular, however, the pressure at the bottom must be set equal to

$$p(\xi)\,\overline{\cos}^2\phi$$

Considering this, we can easily obtain using equation (10a) as a basis

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(QV\alpha) + \frac{AV^2}{cR^{1.314}}\frac{1}{\overline{\cos}\,\phi} = -\frac{dz}{dx}Ag\,\overline{\cos}^2\phi \tag{23}$$

The equation for steady flow can be found from

$$-\frac{dz}{dx}\left(\overline{\cos}^2\phi - \frac{Q^2\alpha s}{gA^3}\right) = \frac{Q\alpha s}{gA^3}\frac{dy}{dx} + \frac{d\alpha}{dx}\frac{Q^2}{A^2g} + \frac{Q^2}{A^2gcR^{1.314}}\frac{1}{\overline{\cos}\,\phi} \tag{24}$$

Equation (16) for computing the backwater and drop-down curves has the following new form:

$$\Delta X = -\frac{\Delta Z\left(\frac{A^2}{Q^2}\overline{\cos}^2\phi - \alpha\frac{S}{gA}\right)}{\frac{\alpha S}{gA}\overline{\tan}\,\phi + \frac{1}{cgR^{1.314}\overline{\cos}\,\phi}} \tag{25}$$

Similar to equation (19), we have

$$\frac{1}{\overline{\cos}\,\phi}V_{cr} = \sqrt{g\frac{A}{\alpha S}} \tag{26}$$

Application to Pipe Lines

The equations derived above are also applicable to pipe lines. The following simplifications are now apparent:

1. A depends uniquely on X and is independent of Z.

2. Q is uniquely a function of the time, t, and is not dependent on X. With this,

$$\frac{\partial Q}{\partial t} = f(t)$$

Equation (23) can now be written

$$\frac{\partial Q}{\partial t}\frac{1}{Ag\overline{\cos}^2\phi} + \frac{Q}{Ag}\left(\alpha\frac{\partial V}{\partial X} + V\frac{\partial \alpha}{\partial X}\right)\frac{1}{\overline{\cos}^2\phi} + \frac{V^2}{cgR^{1.314}\overline{\cos}^3\phi}$$

$$= -\frac{\partial Z}{\partial X}$$

or

$$\frac{\partial Q}{\partial t}\frac{1}{Ag\cos^2\phi}+\frac{\alpha}{g}V\frac{\partial V}{\partial x}\frac{1}{\cos^2\phi}+\frac{V^2}{g}\frac{1}{\cos^2\phi}\left(\frac{\partial a}{\partial x}+\frac{1}{cR^{1.314}\cos\phi}\right) \tag{23a}$$

$$=-\frac{\partial z}{\partial x}$$

The cross-sectional area, a, normal to and the velocity, v, parallel to the axis of the pipe are introduced in place of A and V. Also dx is replaced by an element of the center line of the pipe, dl (figure 5). Then

$$a=A\cos\phi;\quad v=\frac{V}{\cos\phi};\quad dl=\frac{dx}{\cos\phi}$$

Since now V, A, and α and, therefore, the entire left-hand side of equation (23a), are independent of Z, we integrate with respect to X, or rather l, and obtain

$$\frac{dQ}{dt}\int_1^2\frac{dl}{ag}+\alpha\frac{v^2}{2g}\Big|_1^2-\int_1^2\frac{v^2}{2g}\frac{d\alpha}{dl}dl$$

$$+\int_1^2\frac{v^2}{g}\frac{d\alpha}{dl}dl+\int_1^2\frac{v^2\,dl}{2gcR^{1.314}}=-z\Big|_1^2$$

The third and fourth terms are combined and the sum integrated by parts. Thus

$$-\int\frac{v^2}{2g}\frac{d\alpha}{dl}dl+\int\frac{v^2}{g}\frac{d\alpha}{dl}dl=\int\frac{v^2}{2g}\frac{d\alpha}{dl}dl=\frac{1}{2g}\int_1^2 v^2\frac{d\alpha}{dl}dl$$

$$=\frac{1}{2g}\left[v^2\alpha\right]_1^2-\int_1^2\alpha\,2v\frac{dv}{dl}dl \tag{27}$$

This expression represents the loss caused by a change in the velocity distribution.

Since Z represents the sum of the elevation and pressure heads, it may be replaced by

$$z = h_g + h_p$$

where h_g = elevation head measured from any conveniently chosen datum plane and h_p = the pressure head (figure 5).

The final equation is

$$\frac{dQ}{dt}\int_1^2 \frac{dl}{ag} + \frac{\alpha_2 V_2^2}{g} - \frac{\alpha_1 V_1^2}{g} - \frac{1}{2g}\int_1^2 \alpha 2 v \frac{dv}{dl}\, dl + \int_1^2 \frac{v^2}{2g}\frac{dl}{cR^{1.314}} + h_{g_2} - h_{g_1} + h_{p_2} - h_{p_1} = 0 \qquad (28)$$

This expression can be reduced to the ordinary Bernoulli equation by putting $\alpha = 1$, thus

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} + h_{g_2} - h_{g_1} + h_{p_2} - h_{p_1} = 0 \qquad (28a)$$

According to the so-called $\frac{1}{7}$-power law[10], the value of $\alpha = 1.02 = \frac{100}{98}$.

Examples

The application of the computation of backwater curves has been given above. It will now be shown that the weir formula can also be derived from the general equations. It is evident that such a formula will not contain an accurate value of the weir coefficient. Equation (20b), given above, for the case of a rectangular weir is

$$Q = s\mu\sqrt{2gh^3}$$

A triangular weir can be treated in a similar fashion (figure 6). We have

$$gA^3 - Q^2 \alpha s = 0$$

$$A = (D \tan \beta) D = D^2 \tan \beta$$

$$s = 2D \tan \beta$$

$$\frac{V^2}{2g} = \frac{A}{2\alpha s} = \frac{D^2 \tan \beta}{4\alpha D \tan \beta} = \frac{D}{4\alpha}$$

The head upstream from the weir can in practice be replaced by

$$h = D + \frac{V^2}{2g}$$

With this

$$h = D + \frac{D}{4\alpha} = \frac{4\alpha + 1}{4\alpha} D$$

$$D = h \frac{4\alpha}{4\alpha + 1}$$

$$gD^6 \tan^3 \beta = 2Q^2 \alpha D \tan \beta$$

$$Q^2 = \frac{gD^5 \tan^2 \beta}{2\alpha} = \frac{g}{2\alpha} \frac{4^5 \alpha^5}{(4\alpha+1)^5} h^5 \tan^2 \beta$$

To simplify, we place

$$\frac{4^4 \alpha^5}{(4\alpha+1)^5 \alpha} = \mu'^2$$

Then

$$Q = \tan \beta \, \mu' \sqrt{2gh^5}$$

The ordinary form of the equation is

$$Q = \tan\beta \, \frac{8}{15} \, \mu \sqrt{2g} \, h^{5}$$

The value of must be determined experimentally. For a sharp-crested weir with $\beta = 45^{\circ}$, it is 0.59.

Two further examples taken from pipe-line problems are now considered, the first being a sudden expansion in a pipe (figure 7). The loss of head, h_w, can be calculated from equation (27), thus

$$h_w = \frac{1}{2g}\left[v^2 \alpha \Big|_1^2 - \int_1^2 \alpha 2 v \frac{dv}{dl} dl \right]$$

Also

$$Q = \frac{V}{a}$$

At the transition from the smaller pipe to the larger pipe, the flow is similar to an expanding jet in which the velocity, w, would be present. Within this transition $\alpha > 1$ and is easily computed from

$$\alpha = \frac{wQ}{vQ} = \frac{w}{v}$$

In order to integrate, we replace the sudden transition by a very short transitional cone,

Within the limits of this cone, the integral

$$2\int \alpha v \frac{dv}{dl} dl$$

is easily computed, thus

$$\int_1^{2a} \alpha v \frac{dv}{dl} dl = \int_1^{2a} \frac{w}{v} v \frac{dv}{dl} dl = \int_1^{2a} w \frac{dv}{dl} dl$$

On account of the shortness of the transition, it is permissible to place

$$w \approx v_1$$

Then

$$\int_1^{2a} \alpha v \frac{dv}{dl} dl = v_1 \int_1^{2a} dv = v_1(-v_1 + v_{2a})$$

$$v_{2a} = v_2$$

Downstream from cross section Z,

$$v = \text{const.}\ ;\ \frac{dv}{dl} = 0$$

and when this is so, the integral is also equal to zero.

h_w is easily computed now from

$$h_w = \frac{1}{2g}\left[v_2^2 - v_1^2 + (v_1^2 - v_2 v_1)\cdot 2\right]$$

$$= \frac{1}{2g}\left[v_2^2 + v_1^2 - 2 v_a v_1\right] = \frac{1}{2g}(v_1 - v_2)^2$$

Equation (28) is evidently applicable for establishing the influence of the closing time of valves or gates on water hammer, as well as for investigating other nonsteady types of flow in pipe lines. Let the problem be to find Q as a function of the time when the discharge gate at the end of a pipe line is suddenly opened. The trivial solution is obtained that Q rapidly approaches the value

$$Q = f_2 \sqrt{2 g H_g}$$

Since this problem has little practical value, it will be considered only briefly here.

If we assume that α is not subject to any variation throughout the whole length of the pipe line, we have according to equation (28)

$$\frac{dQ}{dt}\int_1^2 \frac{dl}{ag} + Q^2\left[\int_1^2 \frac{dl}{2ga^2cR^{1.314}} + \frac{\alpha}{2ga^2}\right] + (h_{g_2} - h_{g_1}) = 0$$

Position 1 is at the level of the reservoir feeding the pipe line, and position 2 is at the gate at the end of the pipe line. The integrals

$$\int_1^2 \frac{dl}{ag} \quad \text{and} \quad \int_1^2 \frac{dl}{2ga^2cR^{1.314}}$$

can be evaluated for any pipe line to be investigated. A nonhomogeneous differential equation of the second degree is obtained whose solution is best obtained by methods of approximation (Runge Kutta, "Iterationsverfahren." "Method of Iteration.")

This method of solution is to be recommended for accurate investigations because the coefficient of the term, Q^2, is a function of the time if the cross-sectional area, a_2, changes as a consequence of the action of the gate mechanism. In many cases this dependence cannot be represented analytically in a simple way. The most unfavorable case is frequently that in which this cross section decreases from a maximum value of zero in a definite time, T_s. Then when the gate is closed, $Q = 0$. T_s is called the opening or closing time. The difference between the total pressure and that pressure obtained with steady flow is called the dynamic pressure. It amounts to

$$h_{p_{dyn}} = \pm \frac{dQ}{dt}\int_1^2 \frac{dl}{ag}$$

For an approximation, we generally assume that Q decreases linearly from its maximum value to zero in the time, T_s, or

$$Q = Q_{max}\left(1 - \frac{t}{T_s}\right)$$

$$\frac{dQ}{dt} = \frac{Q_{max}}{T_s} \; ; \; h_{p_{dyn.}} = \frac{Q_{max}}{T_s}\int_1^2 \frac{dl}{ag}$$

The relative pressure head is

$$\lambda = \frac{h_{p_{dyn}}}{h_{g_1} - h_{g_2}} = \frac{Q_{max}}{T_s(h_{g_1} - h_{g_2})}\int_1^2 \frac{dl}{ag}$$

If a is constant and L is the total length of the pipe line, it follows that

$$\lambda = \frac{Q_{max}L}{T_s ag(h_{g_1} - h_{g_2})} = \frac{V_{max}L}{T_s g(h_{g_1} - h_{g_2})}$$

In conclusion, it should be mentioned that the basic equation of Allievi proceeds immediately from equation (23), thus

$$\frac{\partial V}{\partial t} = -g\frac{\partial h}{\partial l}$$

providing Allievi's assumptions are included; namely, constant cross-sectional area of the pipe, and frictionless flow.

Summary

Starting with the fundamental equations of dynamics, a formula for the impulse principle was obtained for the slope of the water surface in an open channel as a function of the cross-sectional area of the channel, the bottom slope, the discharge, and, in certain circumstances, the velocity distribution at a cross section. A procedure for computing backwater curves was derived in terms of these factors.

The two types of flow, shooting and streaming, were discussed in terms of this equation after neglecting all losses. Several general conclusions relative to the general types of flow over humps in the bottom of a canal and through contracted canals were also discussed. Next the equation was extended to channels of steep slops. The differential equation obtained thereby can be integrated for closed pipe lines, giving the so-called general Bernoulli equation. At the conclusion several examples were analyzed.

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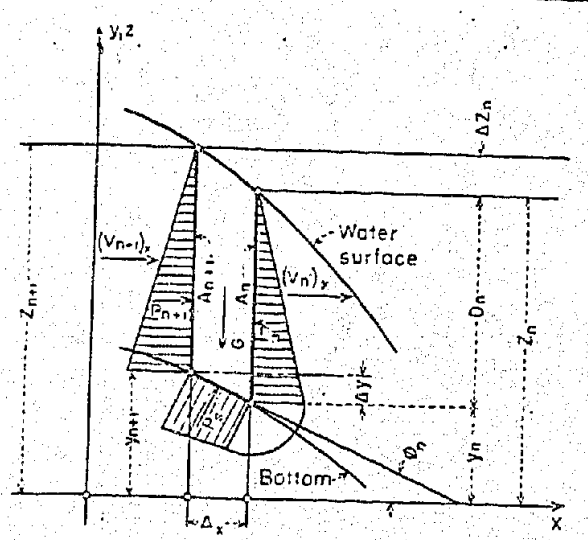
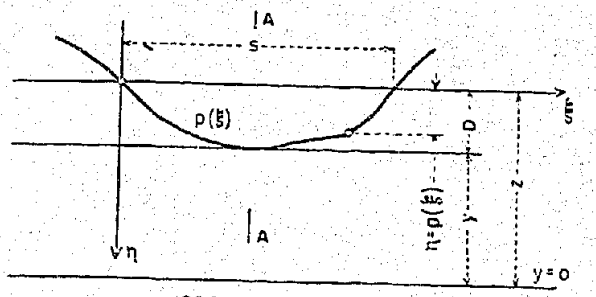


FREE BODY DIAGRAM AT SECTION A-A



CROSS SECTION

FIGURE 1 - LONGITUDINAL AND CROSS SECTIONS OF A CHANNEL

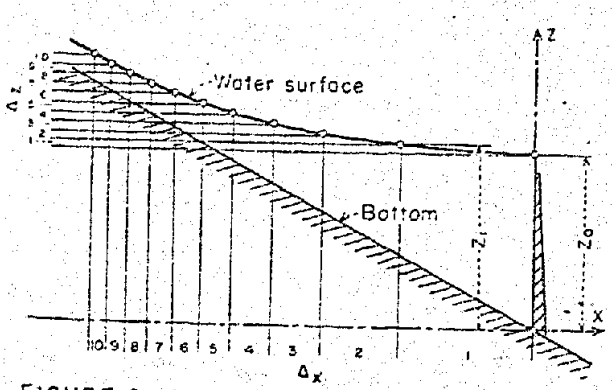


FIGURE 2 - GRAPH OF A BACKWATER CURVE

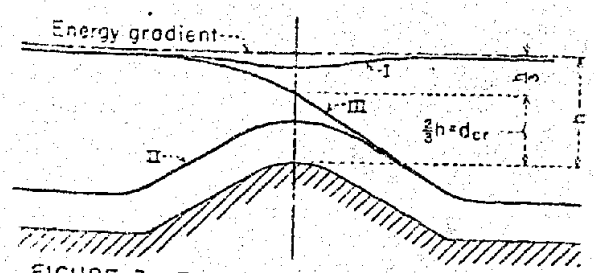


FIGURE 3 - THE THREE POSSIBLE TYPES OF FLOW OVER A HUMP IN THE BOTTOM OF A FLUME

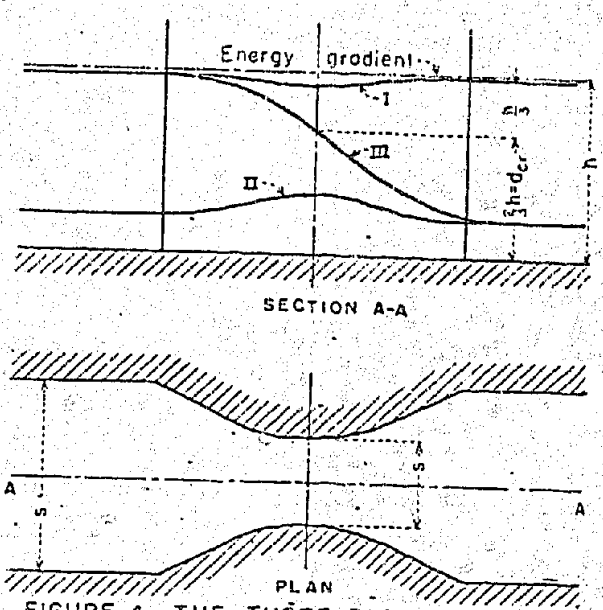


FIGURE 4 - THE THREE POSSIBLE TYPES OF FLOW THROUGH A CONSTRICTION OF A FLUME

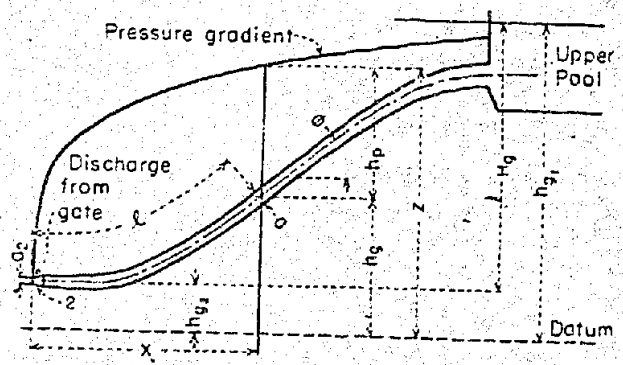


FIGURE 5 - LONGITUDINAL SECTION OF A PIPE

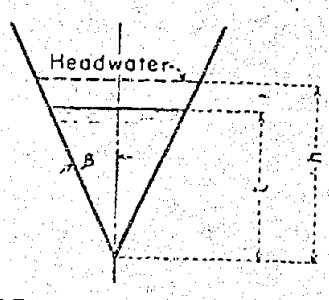


FIGURE 6 - TRIANGULAR WEIR

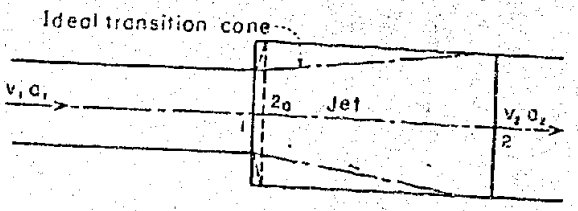


FIGURE 7 - SUDDEN ENLARGEMENT OF A PIPE