UNITED STATES
DEPARTMENT OF THE INTERIOR
BUREAU OF RECLAMATION

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THE PHENOMENA OF AUTO-OSCILLATION IN HYDRAULIC INSTALLATIONS

by Y. ROCARD

- - -

TRANSLATION BY
D. J. HEBERT, ASSOCIATE ENGINEER

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Denver, Colorado,
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A Translation of Pages 5 to 30
of

LES PHENOMENES D'AUTO-OSCILLATION DANS LES
INSTALLATIONS HYDRAULIQUES

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TRANSLATOR'S PREFACE

The following is but a partial translation of the treatise on auto-oscillation; so a translation of the table of contents is also included.

The parts translated are the introduction which presents the author's general theory and Part 1 which treats the application of the theory to the simpler cases.

D. J. Hebert.

Denver, Colorado
July 1, 1943

LES PHENOMENES D'AUTO-OSCILLATION DANS LES INSTALLATIONS HYDRAULIQUES

By Y. Rocard

OUTLINE

INTRODUCTION

The phenomena of pressure rises for varied regime in conduits have appeared to be mysterious for a long time. For the most simple and frequently occurring form, that of water hammer, it seemed to be impossible to give a theoretical explanation in accord with the facts until Allievi in his classical work in 1905 discovered a method of integration for making an analysis.

It is necessary for that which follows to clarify the nature of the problem as resolved by Allievi.

In a conduit of constant cross section with uniform characteristics over its entire length, the movement of the water is established by the following equations.[1]

[1] ρ = density, P = pressure, u = velocity at a point on the abscissa x measured along the conduit.

First, the hydrodynamic equation of motion:

$$(1)\quad \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0.$$

Second, the equation of continuity:

$$(2)\quad \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

The first simplification introduced by Allievi for studying the phenomenon of propagation consisted of neglecting the term $\rho u \frac{\partial u}{\partial x}$ in equation 1. For propagation of a disturbance, the velocity u of the water at each point is effectively a function of $(t \pm \frac{x}{c})$ where c equals the velocity of sound in water. It is clear, then, that:

$$\frac{\partial u}{\partial x} = \frac{1}{c} \frac{\partial u}{\partial t}$$

and that

$$u \frac{\partial u}{\partial x} = \frac{u}{c} \frac{\partial u}{\partial t}.$$

Since c is of the order of 1,400 m./sec. and u is of the order of several meters per second, then by neglecting the term $\frac{u}{c}\frac{\partial u}{\partial t}$ the effect is an error of only several parts per thousand in the term $\frac{\partial u}{\partial t}$. Besides, this is the classical simplification in acoustics where only small movements are studied, so the equations of Allievi:

$$(1') \quad \rho \frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0$$

$$(2') \quad \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0$$

are none other than the fundamental equations of acoustics.[1] These equations should be completed with terms expressing the effects of viscosity, but these will be considered later.

[1] The propagation of sound has for other reasons been capable of analysis by equations (1) and (2). See, for example, "Propagation et absorption du son" fascicule 222 de la collection d'Actualités scientifiques, Hermann éditeur.

Equations (1') and (2') can be solved only if one knows the relation between P and ρ which characterizes the compressibility of the medium. If one considers an adiabatic phenomenon and assumes:

$$c^2 = \left(\frac{dP}{d\rho}\right) \text{adiab.}$$

the general solution of the equation is

$$(3) \quad P = F\left(t - \frac{x}{c}\right) - f\left(t + \frac{x}{c}\right) + P_o$$

$$(4) \quad u = \frac{1}{\rho c}\left[F\left(t - \frac{x}{c}\right) + f\left(t + \frac{x}{c}\right)\right] + u_o$$

F and f are arbitrary functions and c is the speed of sound.

In the problem of acoustics, the functions F and f are determined by the boundary conditions which consist in general of imposing on one of the variables P or u a law of variation with time at two limits

$x = 0$, $x = \ell$ of the conduit.

If, for example, one imagines a pipe where for $x = 0$ one imposes on a piston a given displacement

$$\delta x = a \sin \omega t$$

and where at $x = \ell$ one sets up a solid wall, the result would be

$$u = \frac{\partial \delta x}{\partial t} = a\omega \cos \omega t \quad \text{for} \quad t = 0$$

$$u = 0 \quad \text{for} \quad x = \ell.$$

For these conditions the physical problem of knowing the state of the air at every point is perfectly determined as is also the mathematical problem, since the functions F and f are easily determined, as one can see.

In the problem of water hammer, although handled by the same equations the situation is vastly different. At one of the limits of the conduit there is a condition of an acoustic nature, for example, the surface of a reservoir at constant pressure, $P = 0$ for $x = \ell$; and at the other limit of the conduit one exerts an action upon the discharge by varying a geometric parameter (valve being closed following a certain law, pump or turbine placed in a motion whose effect on the pressure P and the velocity u constitute a new problem for solution).

Let α be this parameter, for example, the degree of opening of a valve, then one has the function $\alpha = \alpha(t)$. One must first find the action of α upon the discharge of water; that is, one must determine a relation:

$$(5) \quad \phi\,(P,\ u,\ t,\ \alpha(t)) = 0 \quad \text{for} \quad x = 0$$

and then integrate the equations (3), (4), with this relation (5) as the condition at the limit for $x = 0$.

The enormous progress attributed to Allievi consisted first of a bold simplification of the relation (5) for cases already widely extended and then a method of iteration which was particularly elegant for obtaining the corresponding solution.

Let us take, for example, the case of a valve which is closed at the end of a conduit: the section for flow at the end is reduced in the ratio α, α being an arbitrary function of time between zero and infinity. If the conduit is completely open, the velocity u is related to P for steady flow by the relation

$$(6) \quad P + \frac{1}{2} \rho u^2 = P_o,$$

or $\frac{1}{2} \rho u^2 = P_o - P_1$, or in a form even more familiar to engineers,

$$(7) \quad u = \sqrt{2gh}$$

h = height of water corresponding to the difference in pressure $P_o - P$; that is, vertical difference in level of the conduit. Allievi assumed simply that the action of the valve reduces the velocity u, based on the total section of the conduit, in the ratio α as follows:

$$u = \alpha(t) \sqrt{2gh}$$

or

$$(8) \quad \frac{1}{2} \rho u^2 = \left[P_o - P\right]\left[\alpha(t)\right]^2$$

h or P cease being constant but vary according to equations (1') and (2').

Expressed another way, if one draws a graph with velocity u at the outlets as abscissa and the pressure $P_o - P$, as ordinate, the hypothesis of Allievi is that at each instant the point representing the regimen at the outlet is located on a known curve derived from a family of curves for the parameter α and fixed by the value that one gives the parameter α at time t. In the particular case of a valve at the outlet of a conduit, the family of characteristic curves comprises the parabolas with the equation

$$P_o' - P = \frac{1}{2}\rho u^2.$$

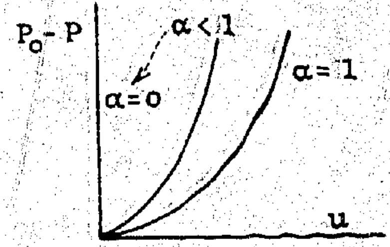

Fig. 1

We will not stress the very elegant solution of equations (1') and (2') furnished by Allievi for such conditions at the limits.

The works of Allievi, already important and numerous in themselves, have given place to a multitude of developments and applications consisting in general of complicating the characteristics of the conduit in accordance with necessity and practice. Noted among others is the work of GADEN on surge tanks and of JAEGER on multiple conduits. One could not do better than to refer to the works of these three authors for gaining an appreciation of the degree to which practical developments have been pushed.

As for the theoretical value of these works, it must be said that they are strictly limited. None of the modern authors writing about the subject have brought this out. We think, on our part, that it is much better to say frankly that we are stumped and that it is bad to present to students and future engineers results acquired to date under an erroneous form which deprives them of any possibility of attempting to perfect the theory in view of the important and real problems posed by practice and which are entirely incompressible if one is bounded by Allievi's approach.

Allievi's solution depends, as we have said, principally on the hypothesis that at each instant the figurative point (in the diagram of fig. 1) of the regimen at the outlet is found on a known curve (varying besides with time in a known manner). In the particular case of a valve which is closed at the end of a conduit, the answer can be obtained only by using for a case of clearly variable flow the equation 6, Bernouilli's theorem, which is valid only for the steady state and in a form which assumes the fluid to be incompressible, an approximation which is admissible here for other reasons.

Strictly speaking, it would be necessary to take from equation (1) the complete integral with respect to x between the limits of the conduit.

which gives:

$$(9) \quad \rho \int_0^l \frac{\partial u}{\partial t}\, dx + \frac{1}{2}\rho u^2_{(x=0)} = P_o - P.$$

It is evident then that in varied flow the relation between the velocity and the pressure at the outlet of the conduit can not be expressed by the restriction of a figurative point on a curve in the diagram $P_o - P, u$; but the value of discharge at each instant depends upon the pressure at the outlet <u>and also upon the acceleration</u> $\frac{\partial u}{\partial t}$ <u>of the water at each point of the conduit</u>. In addition, in the valve itself, during the process of decreasing the section we will find values of $\frac{\partial u}{\partial t}$ of considerable magnitude at each point, and it is clear that the velocity u referred to the total section of the conduit can not follow a law as simple as (8) and can no longer be expressed for a given value of closure α by a simple relation between u and P. It is only by passing over all these difficulties that one can simplify hydrodynamics sufficiently to make it into hydraulics, and particularly into the theory of water hammer. From this point on it seems to us to be a matter of entirely conventional work. It would be necessary each time to examine carefully the character more or less constant with reality, of the characteristic curves adopted for the outlet device (such as the parabolas $\frac{1}{2}\rho u^2 = (P_o - P)\alpha^2$ for closure of a valve) before adopting any conclusions.

Let us hasten to say that in the majority of cases this discussion shows that for transition flows which are quickly provoked and allowed to evolve by themselves, Allievi's conception is largely sufficient to account for the facts in a nearly quantitative manner.

The theory developed on the preceding basis has found its most elegant expression and most usable form in practice in the work of M. L. Bergeron, who has developed a complete graphical interpretation of the solutions of equations (1') and (2') with boundary conditions of the Allievi type. The most recent form is in La Technique Moderne, January 15 and February 1, 1936, pp. 33 and 75, respectively. The graphical method of M. L.

Bergeron constitutes not only a graphical interpretation of the computations but also a true method of integration which supplies precise and exact solutions of equations (1') and (2') even in the case where the system of characteristic curves of Allievi have an analytical form which is too complicated for a computed solution.

In other words, the graphical method of M. Bergeron gives the maximum results which can be obtained in depending on the framework of the hypothesis of Allievi and it constitutes the happiest and most complete expression of what might be called the theory of Allievi. Its validity is always strictly limited to that of Allievi's hypothesis and it requires, as we have seen, that a characteristic relation between pressure and discharge be known for the outlet at each instant.

IDEA OF HYDRAULIC IMPEDANCE - ELECTRIC ANALOGY

To insert at some point in a conduit a device which imposes a relation between pressure and discharge amounts to imposing an hydraulic *impedance* at this point by analogy with the notion of electrical impedance. The pressure at each point is in effect the analogue of voltage and the discharge the analogue of electrical current. The controlled opening at the outlet of a conduit sets up a velocity u, which is related to the driving pressure P_o by the relation

$$P_o = \frac{1}{2} \rho u^2$$

which has for small changes in u the characteristics of a resistance. If Q is the discharge uS, where S is the section one has, in effect,

$$(10) \qquad \delta P = \rho u \delta u = (\rho \frac{u}{S}) \delta (Su) = \rho \frac{u}{S} \delta Q.$$

This is the analogy of $E = Ri$, the resistance R being in this case $\frac{\rho}{S} u$. The closure of a valve at the end of a conduit amounts to varying the resistance R at the end of a long line fed by a battery E, and the problem resolved by Allievi has for its counterpart the problem of the variation of an electric flow in a line supposedly endowed with inductance and capacitance per unit of length. The equations for propagation in such an electric line are obtained as follows:

Fig. 2

Let ℓ and γ be the inductance and the capacitance per unit of length. For the length dx one has the inductance ℓ dx and the law of inductance gives

$$dE = -\ell\, dx \frac{di}{dt}$$

If one considers the voltage E and the current i as functions of x and time, then we can write

$$(11) \qquad \frac{\delta E}{\delta x} + \ell \frac{\delta i}{\delta t} = 0$$

(this is analogous to $\rho \frac{\delta u}{\delta t} + \frac{\delta P}{\delta x} = 0$).

In addition, the capacitance γdx of the element dx involves a variation of i such that the charge $\left[i_{(x+dx)} - i_{(x)}\right]dt$ accumulated in the element during time dt is equal to γdx when multiplied by the voltage dE at the boundaries of the element thus

$$(12) \qquad \frac{\delta E}{\delta t} + \frac{1}{\gamma} \frac{\delta i}{\delta x} = 0$$ (analogous to the equation of continuity.)

Equation (11) is analogous to equation (1') and equation (12) is analogous to equation (2'), and the general solution of equations (11) and (12) is:

$$E = F\left(t - \frac{x}{c}\right) - f\left(t + \frac{x}{c}\right)$$

$$i = \sqrt{\frac{\gamma}{\ell}}\left[F\left(t - \frac{x}{c}\right) + f\left(t + \frac{x}{c}\right)\right]$$

if one supposes $c = \frac{1}{\sqrt{\ell\gamma}}$, the quantity that was found to be equal to the speed of light in a vacuum when computing the inductance ℓ and the capacitance γ.

To simplify the case let us imagine a single wave being propagated in the positive sense: f is equal to zero, and we have the relation

$$F\left(t - \frac{x}{c}\right) = E = \sqrt{\frac{\ell}{\gamma}}\, i$$

The term $\left(\sqrt{\frac{\ell}{\gamma}}\right)$ is called the iterative <u>impedance</u> of the line.

Coming back to one conduit and particularly to equations (3) and (4), let us consider also the case of a single wave in a positive sense. We have

$$P = \rho c u$$

or introducing the discharge Q

$$P = \frac{\rho c}{S} Q$$

$\frac{\rho c}{S}$ will be the <u>iterative impedance</u> of the conduit and has the character of <u>pure resistance</u> which means that at each point if the conduit is traversed by one or more waves in a unique sense, the variations of pressures are in phase with the variations of velocity.

However, all hydraulic impedances are not pure resistance. Let us place, for example, at the top of a conduit a piston weighted by a mass m. If u is the velocity of the water at the level of the piston, the displacement of the piston is

$$\int u\, dt$$

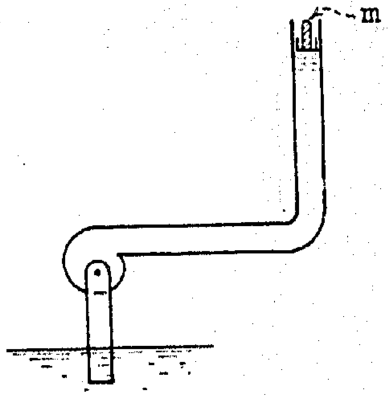

Fig. 3

The force $F = m \frac{d^2x}{dt^2}$ is set up due to the interia of the mass, which comes back to imposing a pressure $\frac{P}{S}$ equal to F and given by

$$P = \frac{m}{S} \frac{du}{dt}$$

or introducing the discharge

$$P = \frac{m}{S^2} \frac{dQ}{dt}$$

$\frac{m}{S^2}$ is then an impedance which makes the pressure proportional to <u>rate of change</u> of the discharge. By analogy with $E = L \frac{di}{dt}$, this is an hydraulic inductance.

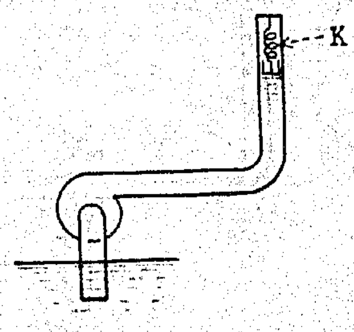

Fig. 4

Let us now close the upper end of the conduit and maintain the piston with a spring K which could be an air reservoir. The replacement of the piston $\int u\, dt$ sets up a force which is

$$F = K \int u\, dt$$

where K = stiffness of spring,

or

$$P = \frac{K}{S^2} \int Q\, dt$$

By analogy with the law $E = \frac{1}{C} \int i\, dt$ which gives the "tension" at the limits of a condenser being charged, we can say that $\frac{K}{S^2}$ is an hydraulic capacitance.

But in a number of practical cases as, for example, one closes a heavy vane with interposition of elastic mechanisms, there is introduced upon the conduits these mechanisms which have the character of hydraulic inductance or capacitance.

In the water-hammer theory as known, whether it concerns Allievi's computations or their graphical expression by M. Bergeron, these cases are completely put aside because they cannot be treated.

Allievi's theory reduces itself then to the case where all the hydraulic impedances are pure resistances which can be a function of discharge as the example of the valve previously treated has shown it to be.[1]

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[1] In electricity there are also nonlinear resistances whose value depends upon the current which is passing. But there is no electrical means to represent exactly hydraulic resistances proportional to discharge.

In a form more familiar to engineers, we could say that Allievi's theory is limited to the case where the machines at the end of a conduit have a function which can be defined by characteristic curves of pressure vs. discharge, but any time that these machines present inertia and elasticity this cannot be done and, it is not legitimate to admit without discussion, as everyone has done up till now, that for a machine as complicated as a pump or a turbine, the functioning in variable regimen does not depart from the static characteristics of pressure discharge.

AIM OF THE PRESENT WORK

This work is designed to lead up to even more complex phenomena, such as when a conduit is coupled to a mechanical system susceptible to movement and whose displacements provoke in return a reaction upon the flow of fluid. Such a system under certain conditions can start oscillation <u>by itself</u>, as is well known, and the classical theory of water hammer cannot in any manner explain the birth of such oscillations nor predict what will happen to them. We propose to find these conditions of oscillations and describe the oscillatory regimen which results. An example of such a system is furnished by an organ pipe provided with a beating reed A. Under the influence of a permanent jet of air the reed is made to vibrate in resonance, usually not well known, with the period of the pipe.

In the first part we will develop the study of oscillations for a simple conduit; in the second part we will extend the results to the case of complex conduits by regularly using the notion of hydraulic impedance.

In addition, we will present important enough results touching on the resistance to radiation offered by a reservoir, the absorption of oscillatory energy due to loss of head in a conduit, the determination of the dynamic characteristics of machines, such as centrifugal pumps, characteristics valid even in varied regimen, etc.

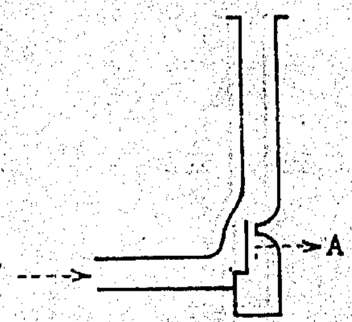

Fig. 5

PART 1

AUTO-OSCILLATION OF WATER IN A CONDUIT TERMINATED BY A MACHINE OF KNOWN CHARACTERISTICS WHICH CONTAINS A MECHANISM CAPABLE OF VIBRATING AND MODULATING THE DISCHARGE

We will discuss a particular practical case in order to show the relation between the water in the conduit and the movable mechanism at the end. Model of figure 6 represents a known centrifugal pump in which the discharge is regulated by moving the front part A of the directing vanes. In other words, there is a ring of movable directing vanes in front of a ring of fixed directing vanes. These movable blades can be oriented by the action of a mechanism which moves each of them about its axis and more or less closes the discharge section.

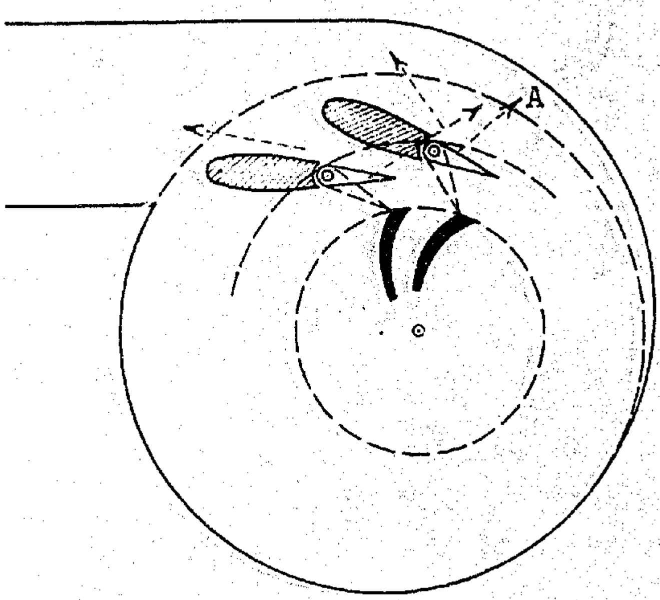

Fig. 6

Let α be the degree of the opening of the vanes, $\alpha = 1$ give the maximum discharge, and $\alpha = 0$ represent closure of the pump. The static characteristics of such a pump are known for each degree of closure, α. They take the following form (fig. 7):

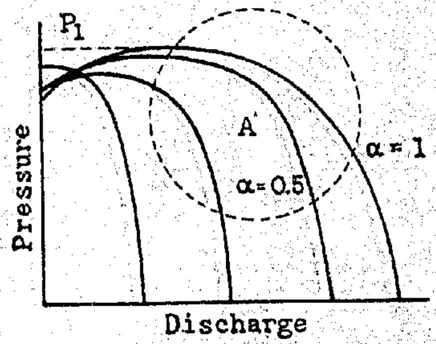

Fig. 7

It can be seen that in the region of operation A when α increases at constant discharge, the pressure increases; and, when the discharge increases with constant opening, the pressure decreases. For small changes in α and in the velocity of the water u in the conduit, the characteristic of the pump reduces to

$$\delta P = \gamma \delta \alpha - \eta \delta u \quad \text{...} (13)$$

making the assumption [Allievi type] that in variable regime the static characteristics of the pump are valid and that alternate variations in α and in u produce variations of P in phase with themselves.

This simplifying assumption can be made here without upsetting the principle for explaining the priming of oscillations.

The hypothesis is now introduced that the control of the movable directing vanes is not rigid and that the latter are susceptible to movement under the influence of changes in the pressures exerted on them.

Let us formulate from these conditions the equation of possible movement of the vanes.

Let Θ be the angle between the actual position of the vanes and the position of repose (corresponding to an equilibrium condition of the control mechanism). The vanes can vibrate around this position with a movement according to the law

$$i' \frac{d^2\Theta}{dt^2} + k'\Theta = S\,x\,(P_1 - P_2) \quad \text{.................................} (14)$$

where

- i' - is the inertia coefficient, taking into account all that enters into vibration movement in the vane system,
- k' - is the coefficient of rigidity, taking account of all the elastic forces which oppose the flexure of the vanes,
- S - is the area of vanes,
- $P_1 - P_2$ - is the difference in pressure exerted on the two faces of the vane,
- x - is the lever arm of the corresponding force referred to the axis of rotation of the vanes.

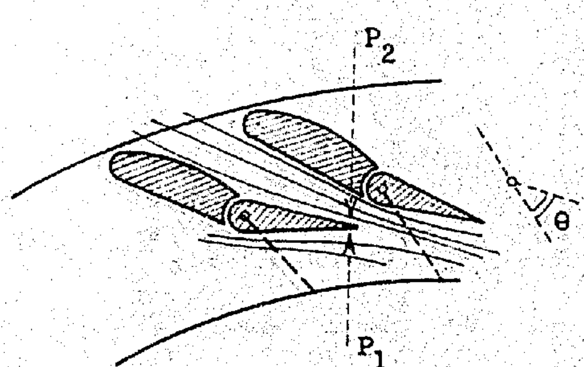

Fig. 8

Instead of the variable Θ we introduce the degree of opening a of the distributor (a varies inversely as Θ). Changing the equation to variable δa from a, we have

$$i' \frac{d^2 \delta \alpha}{dt^2} + k \delta \alpha = - S x (P_1 - P_2)$$

To evaluate the pressures and the variations of pressure, let us take the complete hydrodynamic equation, which is usable even in variable flow.

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

Integrating along a fluid filament in the pump, over a length x,

$$P + \frac{1}{2} \rho u^2 + \rho \bar{x} \frac{\overline{\partial u}}{\partial t} = \text{constant},$$

$\bar{x}$ being a certain mean taken along the chosen length x.

In carrying out this operation more precisely along a movable vane, one sees that P, being at the point of the vane, P_2 is also the pressure at the articulation of the same vane from whence:

$$P_1 - P_2 + \frac{1}{2} \rho (u_1^2 - u_2^2) + \rho x \overline{\frac{\partial u}{\partial t}} = 0$$

$\frac{1}{2} \rho (u_2^2 - u_1^2)$ is then the measure of $P_1 - P_2$ in permanent regime. In oscillating regime it is necessary to take into account that which superimposes on $P_1 - P_2$ in addition to the variations of $\frac{1}{2} \rho (u_2^2 - u_1^2)$, the new term $- \rho x \frac{\overline{\partial u}}{\partial t}$.

Because the distributor, if it functions well, passes at constant pressure a discharge proportional to α, and since the section is constricted in proportion to α, the velocity in the constricted section can be considered constant. On the side P_2, $\frac{\partial u}{\partial t}$ is negligible, while on the side where P_1 is exerted one has $\frac{\partial u}{\partial t} = \frac{u}{\alpha} \frac{\partial \delta \alpha}{\partial t}$. Letting $\xi = \frac{P_1 - P_2}{P}$, the equation becomes

$$i \frac{d^2 \delta \alpha}{dt^2} + k \delta \alpha = (- \xi S x) \delta P + (S x^2 \rho \frac{u}{\alpha}) \frac{d}{dt} \delta \alpha$$

Let

$$S \rho x^2 \frac{u}{\alpha} = f$$

then

$$\left[i \frac{d^2}{dt^2} - f \frac{d}{dt} + k \right] \delta\alpha + (\xi S x)\, \delta P = 0$$

But we have admitted that at the pump the variations of pressure are related to the variations of velocity and of opening by $\delta P = \gamma \delta\alpha - \eta \delta u$. Substituting this value of δP in the preceding equation and changing the significance of the constant k and setting $(\gamma S x)\eta = +\beta$,

$$\left[i \frac{d^2}{dt^2} - f \frac{d}{dt} + k \right] \delta\alpha - \beta \delta u = 0 \quad \text{.........................(15)}$$

The coefficient f with a minus sign indicates a negative damping in the movement of α; it is this which will, in general, be responsible for the beginning of oscillations.

Let us imagine now that our pump discharges through a conduit of length ℓ under a constant water level. One has the following relations in the conduit:

$$P = P_o + F(t - \frac{x}{c}) - f(t + \frac{x}{c})$$

$$u = u_o + \frac{1}{\rho c} \left[F(t - \frac{x}{c}) + f(t + \frac{x}{c}) \right]$$

with the following condition: $P = P_o$ for $x = \ell$.

Then

$$f(t) \equiv F(t - \frac{2\ell}{c})$$

and

$$f(t + \frac{x}{c}) \equiv F(t - \frac{2\ell - x}{c})$$

In particular we will have at the end of the conduit for $x = 0$, where the pump is situated:

$$\delta P = \gamma \delta \alpha - \eta \delta u = F(t) - F(t - \frac{2l}{c}) \qquad (16)$$

$$\delta u = \frac{1}{\rho c} \left[F(t) + F(t - \frac{2l}{c}) \right] \qquad (17)$$

$$\left[\frac{id^2}{dt^2} - f \frac{d}{dt} + k \right] \delta \alpha - \beta \delta u = 0 \qquad (18)$$

Equations (16) and (17) express the relation between the pressure and the velocity which the presence of the conduit imposes on the pump; equation (18) expresses the dynamic relation according to which the velocity of the water reacts on the vanage system of the pump.

One can see here the difference between the problem set down and that of Allievi. Instead of α being an arbitrary function of time, α is a new variable, subject to oscillations and accounted for by a differential equation with the terms of coupling to the other variables, velocity, or pressure of water. Yet in the derivations of our equations we have used the assumption that the dynamic phenomena in the pump are representable by the system of static characteristics. But the assemblage of equations (16), (17), and (18) present, in a simplified form, the properties required for rendering account of the phenomena of auto-oscillation which interest us.

If we consider the coefficients γ, η, β, f, and k to be constant, although in reality they depend on u, equations (16), (17), and (18) are linear and we obtain easily the information about the different regimes which they represent.

Let us try to find the condition for which they have sinusoidal solutions. The most simple are:

$$F(t) + F(t - \frac{2l}{c}) = 2 \sin \omega t \cos (\frac{\omega l}{c})$$
$$\text{.......................(19)}$$
$$F(t) - F(t - \frac{2l}{c}) = 2 \cos \omega t \sin (\frac{\omega l}{c})$$

Equation (16) then gives

$$\gamma \delta a = \frac{\eta}{\rho c} \left[2 \sin \omega t \cos \frac{\omega l}{c} \right.$$

$$\left. + 2 \cos \omega t \sin \frac{\omega l}{c} \right]$$

Putting this value in equation (18), one has

$$\left[i \frac{d^2}{dt^2} - f \frac{d}{dt} + k \right] \left[\frac{\eta}{\rho c} (\sin \omega t \cos \frac{\omega l}{c}) + \cos \omega t \sin \frac{\omega l}{c} \right]$$

$$- \frac{\beta \gamma}{\rho c} (\sin \omega t \cos \frac{\omega l}{c}) \equiv 0 \quad \text{.......................(20)}$$

One can see that if the pump has a flat characteristic (P = constant whatever the discharge), $\eta = 0$, $\gamma = 0$, one has

$$\left[i \frac{d^2}{dt^2} - f \frac{d}{dt} + k \right] \cos \omega t \sin \frac{\omega l}{c} = 0$$

This relation can be satisfied only in the following cases:

1. $\sin \frac{\omega l}{c} = 0$, the conduit vibrates on half waves or on the harmonics $\xi = \frac{2l}{1}, \frac{2l}{2}, \frac{2l}{3}$, etc.

2. $f = 0$, and in addition, $\omega^2 = \frac{k}{i}$. This means that the vanage mechanism introduces neither positive nor negative damping and that the water in the conduit oscillates in the frequency proper of the system of vanage defined precisely by $\omega^2 = \frac{k}{i}$.

These cases offer no practical interest because a pump can never have a flat characteristic independent of discharge, even in variable flow, and besides, there is no means of annuling the coefficient f.

In the general case one has, in developing equation (20), a condition for the frequency by equating to zero the coefficient of cos ωt, thus

$$(k - i\omega^2)\sin\left(\frac{\omega l}{c}\right) - f\,\frac{\eta\omega}{\rho c}\cos\left(\frac{\omega l}{c}\right) = 0$$

and a condition of maintenance by equating to zero the coefficient of sin ωt, thus

$$\left[\frac{\eta(k - i\omega^2) - \beta\gamma}{\rho c}\right]\cos\left(\frac{\omega l}{c}\right) + f\omega\sin\left(\frac{\omega l}{c}\right) \geq 0 \;\ldots\ldots\; (22)$$

In this last equation the sign equals zero is the condition realised for permanence of purely sinusoidal oscillations, but the sign ≥ 0 is the condition which expresses negative damping and, thus, the possibility of the automatic birth of oscillations as a result of any initial disturbance whatever are infinitely small.

In reality the question of the birth of oscillation calls for a more searching examination. Let us imagine that at time $t < 0$, the system be in equilibrium: $\delta P = \delta u = \delta a = 0$ and that at time $t = 0$ the system of equations (16), (17), and (18) becomes suddenly valid. One can imagine that the vane system is blocked by an accessory gear which prevents its vibration previous to $t = 0$. There is a first period from $t = 0$ to $t = \frac{2l}{c}$ when $F(t - \frac{2l}{c})$ equals $F(t)$, for t going from $-\frac{2l}{c}$ to zero is still nothing. One examines the system obtained in placing $F(t - \frac{2l}{c}) \equiv 0$, $F(t) = \delta P$ which gives:

$$\delta u = \frac{1}{\rho c}\,\delta P$$

$$\delta P\left[1 + \frac{\eta}{\rho c}\right] = \gamma\,\delta\alpha$$

$$\left[i\,\frac{d^2}{dt^2} - f\,\frac{d}{dt} + k\right]\delta\alpha - \frac{\beta\gamma}{\rho c + \eta}\,\delta\alpha = 0$$

One sees that the equation in $\delta\alpha$ is still of the oscillatory type, that it contains a negative damping term $\left(-f\,\frac{d\,\delta\alpha}{dt}\right)$, and that as a result $f > 0$ is the only condition for the birth of oscillations.

Going back to general equations (21), (22): equation (21), where ω is the unknown, determines a discrete series of frequencies, ω_1, ω_2, ω_3, and so forth, which are the frequencies proper of the conduit system – pump versus oscillating. For those frequencies which give a positive sign to the first term of (21), oscillations are possible. As is well known, it is never possible to establish simultaneously several oscillations at different frequencies. In reality the coefficients and notably β, γ, η, and f are not constants. They depend on the portion of the characteristics used and consequently they vary with the amplitude of the oscillations. This in turn establishes itself in such a manner that a kind of mean value of damping is zero. All these phenomena can be demonstrated in a perfectly clear fashion in the oscillation problem of triode lamps with characteristic curves. Here it is the same thing, with the small difference that the problem of propagation in a conduit becomes complicated.

Taking into account equation (21), the inequality (22) becomes:

$$\left[\frac{\eta(k - i\omega^2) - \beta\gamma}{\rho c} + \frac{\eta f^2 \omega^2}{\rho c\,(k - i\omega^2)}\right]\cos\frac{\omega l}{c} \geq 0$$

or dividing by $\frac{\eta}{f\omega\rho c}$ which is positive

$$\left[\frac{k - i\omega^2}{f\omega} + \frac{f\omega}{k - i\omega^2} - \frac{\beta\gamma}{\eta f\omega}\right] \cos\frac{\omega l}{c} \geq 0 \quad \ldots\ldots\ldots\ldots\ldots(23)$$

but $\frac{f\omega}{k - i\omega^2}$ is equal to $\frac{\rho c}{\eta} \tan\frac{\omega l}{c}$ from which finally

$$\left[\frac{\rho c}{\eta} \tan\frac{\omega l}{c} + \frac{\eta}{\rho c}\frac{1}{\tan\frac{\omega l}{c}} - \frac{\beta\gamma}{\eta f\omega}\right] \cos\frac{\omega l}{c} \geq 0 \ldots(23')$$

We can go no further with the discussion without assuming numerical values. We will take, for example, values which refer to an installation of very great capacity which we have investigated for auto-oscillation.

The moment of inertia of the vanage system, referred to the variable α, is 10^{10} c.g.s. The frequency proper of the vanage system, accounting for the elastic properties of all the control mechanism and linkage, is of the order of 5.5 per sec. This conduit has $k = 1.25 \times 10^{13}$. The velocity of sound in water is 1000 m/s approximately, assuming $\rho c = 10^5$ c.g.s. Finally, f is taken as 5×10^9 c.g.s., and, in addition, the pump characteristics

$$\beta = 3.2 \times 10^9 \frac{\xi}{\alpha}$$

$$\gamma = \frac{1.25}{\alpha} 10^7$$

$$\eta = \frac{10^5}{\alpha}$$

It is not possible to show in this work how numerical determination of these constants was made. We ask the reader to take them as an example for the discussion.

For these conditions, the characteristic frequencies are given by the equation

$$\tan \frac{\omega \ell}{c} = \frac{\eta}{\rho c} \, \frac{f\omega}{k - l\omega^2}$$

$$= \frac{1}{a} \, \frac{5 \times 10^9 \omega}{9.5 \times 10^9 \, (10^3 - \omega^2)}$$

$$= \frac{0.5}{a} \, \frac{\omega}{1.3 \times 10^3 - \omega^2}$$

Let us study more particularly what happens for small values of vane opening and particularly $a = \frac{1}{5}$.

$$\tan \frac{\omega \ell}{c} = \frac{2.5\omega}{1.3 \times 10^3 - \omega^2}$$

If we represent graphically, for a certain value of ℓ chosen for simplicity as 1000 m. or 10^2 c.g.s., on one hand $\tan \frac{\omega \ell}{c}$ as a function of ω, on the other hand $\frac{2.5\omega}{1300 - \omega^2}$ as a function of ω, the intersection of these curves gives us the possible values of ω.

We see that for small values of ω, frequencies distinctly less than 5.55 per second, the value of ω remains in the neighborhood of π, 2π, 3π ..... with $\tan \frac{\omega \ell}{c}$ in the neighborhood of zero.

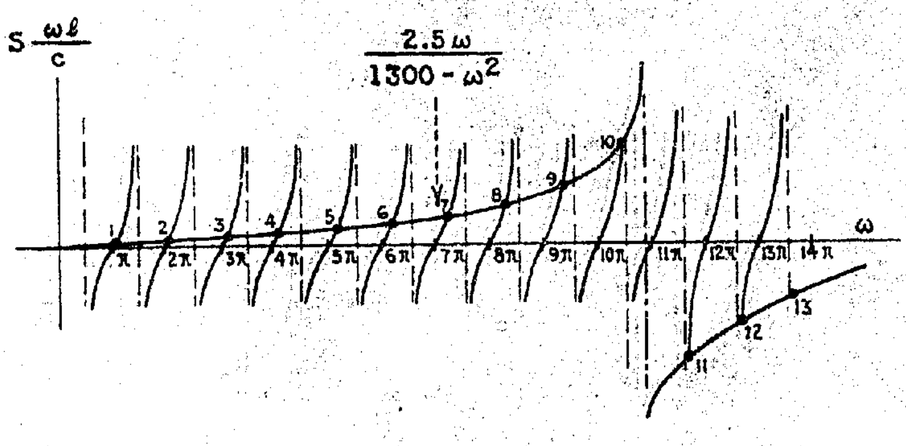

Fig. 9

On the contrary, for ω approaching $5.55 \times 2\pi$, the value of ω establishes itself around 10.5π or 11.5π with $\tan \frac{\omega l}{c}$ approaching infinity. There are other solutions with values of ω distinctly higher, such that $\tan \frac{\omega l}{c}$ approaches zero again.

Now, then, in the system described, with the chosen numerical values, the oscillations can be of two types; first, with a characteristic frequency giving $\tan \frac{\omega l}{c}$ around zero, that is to say, causing the conduit to vibrate on a half wave or on the harmonics $\xi = 2l, \frac{2l}{2}, \frac{2l}{3}$, and so forth; second, with a characteristic frequency near resonance of the vanage, making the conduit vibrate very near to a certain harmonic in the series $\xi = 4l, \frac{4l}{3}, \frac{4l}{5}$, and so forth.

The state of pressures and velocities at any point on the conduit, in the stationary regimen, and in particular at the pump, result from the relations of Allievi and, in particular, from formulas (19).

The pressure P at the pump, setting $F(t) - F(t - \frac{2l}{c}) = 2 \cos \omega t \sin \frac{\omega l}{c}$, takes a small alternating value for oscillations of the first type ($\tan \frac{\omega l}{c} \sim 0$) and one finds in the conduit places where the pressure is very much higher.

On the contrary, for oscillations of the second type with the conduit vibrating on a frequency close to resonance of the vanage, the inverse takes place, and one finds the maximum alternating amplitude of the pressures at the end of the conduit, that is, at the level of the pump.

It remains to be seen, by means of a discussion of equation (23'), what frequency can most effectively establish itself. In this equation and in the case which concerns us, $\frac{\beta \gamma}{\eta f \omega}$ is approximately $\frac{100\xi}{\alpha\omega}$; ξ is a factor of the order of 1/5 to 1/2; α of the order of 1/5 in the example which has been selected, from which we get an order of magnitude of $\frac{100}{\omega}$. For the basic characteristic frequencies, $\omega \sim 2\pi$. For the characteristic frequencies in the neighborhood of resonance of the conduit, $\omega \sim 2\pi \times 5.5 = 35$.

For frequencies of the first type, $\tan \frac{\omega l}{c} \sim 0.23$ reduces to

$$\left[\frac{1}{\alpha} \frac{1}{\tan \frac{\omega l}{c}} - 15\right] \cos \frac{\omega l}{c} \geq 0$$

With $\alpha = 1/5$, we get finally

$$5 \times \frac{1 - 3 \tan \frac{\omega l}{c}}{\tan \frac{\omega l}{c}} \cos \frac{\omega l}{c} \geq 0$$

The frequency which must establish itself can be determined rigorously, as we have said, only by a study of the nonlinear equations which exactly govern the system; but it is evident that it is whatever will have the property of giving the greatest absolute value to the preceding expression which will impose itself. Examination of figure 9, which shows the determination of the roots for ω proves that it is the lowest frequency, $\xi = 2l$, which must establish itself, among those of the first group, but we do not yet know whether or not one of the second group will impose itself. We see that all the points, 5, 6, 7 ..... are prohibited because $1 - 3 \tan \frac{\omega l}{c}$ for these points becomes less than zero.

For frequencies of the second type the inequality (23') becomes numerically

$$\left[\frac{1}{5} \tan \frac{\omega l}{c} + \frac{5}{\tan \frac{\omega l}{c}} - 3\right] \cos \frac{\omega l}{c} \geq 0$$

If one refers to figure 9, one sees that for a frequency of the second group just below resonance of the vanage, $\tan \frac{\omega l}{c}$ has a positive value as well as $\cos \frac{\omega l}{c}$. On the contrary, for a frequency just before resonance (point 9), $\tan \frac{\omega l}{c}$ has a high and positive value, but $\cos \frac{\omega l}{c}$ becomes negative. For frequencies above and beyond, point 11 is rejected for the inverse reason that point 9 was rejected

$(\tan \frac{\omega l}{c} < 0$, but $\cos \frac{\omega l}{c} > 0)$.

Summing up, with the numerical values selected, only the frequencies corresponding to points (10) for the second group and (1) for the first group are to be considered.

As for knowing whether frequency (1) or (10) will establish itself, this depends very much on the numerical values of the different parameters. It is clear that if the characteristic frequency of the vanage (5.5 here) falls very near a harmonique of the series $\xi = 4l$, $\frac{4l}{3}$, $\frac{4l}{5}$, ... it is the frequency of the second group that will maintain. The vanage imposes its frequency on the conduit, which then transmits to all points a rapid vibration of the water. But the inverse can take place, in which case the conduit imposes its frequency on the vanage which takes the movement required for allowing the conduit to vibrate near a half wave.

Note concerning the phenomena in organ pipes

Organ pipes are supplied at their bases with a vibrating reed whose movement can modulate the air being blown in at the base of the pipe. The system comprised by the blower-reed-pipe is evidently governed by equations of the same sort as those we have developed for pumps; so our work will furnish a theory for the functioning of organ pipes if one applies himself to the necessary transposition. An important difference is contained in the fact that one seeks to place the organ pipe in vibration by realizing an accord between the reed and the pipe. This accord is necessary because the air, used as the fluid instead of water, supplies a reaction which is much more feeble (700 times as feeble because of the ratio of densities). So, in the case of an organ pipe, one always has vibrations of the second type.

Note concerning the possible resonance with an imposed frequency

There is to be considered the case where an imposed frequency exists in some part of the system susceptible to oscillation. In the problem set up there might be, for example, a slight unbalance of one of the pumps or of one of the driving motors. For example, the pumps of the installation that we have in mind turned at 4.58 r.p.s. This frequency is quite close to the frequency 5.55 found for the oscillating of the vanage system. But this characteristic frequency of the vanage was calculated for the system vibrating only in air; in the presence of water it has an added mass due to the entrainment of water by the parts that vibrate, inertia increases, and the characteristic frequency diminishes. Thus for a certain stretched membrane one calculates a characteristic frequency of 800 per second, then, in the presence of water, it falls to 160 per second. The added mass is expressed as the inertia multiplied by $(\frac{800}{160})^2 = 25$. In the case at hand where the moment of inertia of the movable vanes is one-twentieth, approximately, of the total inertia of the system with the controlling mechanism, the increase due to the water affects only this one-twentieth part. Thus, the fact that the vibrating vanes do not extend into an undefined fluid but only into the liquid jet which traverses the pump forces us to treat a problem in two dimensions where one finds $\frac{25}{2\pi}$ to be the increase of inertia. The relative increase in our problem will be $1 + (\frac{1}{20} \times \frac{25}{2\pi})$ or approximately $1 + \frac{4}{20}$ and the decrease of frequency will vary as the inverse of the square root which is approximately $1 - \frac{2}{20}$, which gives a characteristic frequency of $5.55 \left[1 - \frac{2}{20}\right] = 5.0$ in the presence of water. Under these conditions, catching up on the frequency 4.58 becomes quite possible. This frequency 4.58 can introduce itself into the system if, for example, a pump has a slight unbalance or a slight eccentricity causing a modulation corresponding to the discharge. It is then practically certain that in the numerical example which we have chosen, a mechanical vibration of pumps, turning at their normal speed, serves to attach itself to the coupled vibrations of vanage and conduit at a frequency of the order of 4.5 per second, the effects of which may be considerable on the parts which have no means of support.

On the contrary, if something happens which increases the elasticity of the vanage, for example, a rupture of the gears, then the coefficient k in our equations becomes very small, the characteristic vibration of the second group has its frequency lowered to that of the fundamental of the conduit considered as an organ pipe, and the resonance which follows produces its maximum effects with regard to the amplitude of the oscillations.

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