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VELOCITY DISTRIBUTIONS AND THE HYDRAULIC DESIGN OF SIDE-CHANNEL INTAKES AND SPILLWAYS, AND TAILRACES
(CONTRIBUTION TO THE STUDY OF FLOWING WATER)

by HENRY FAVRE, DOCTOR OF SCIENCE

TRANSLATION FROM FRENCH
By D. C. McCONAUGHY, ENGINEER

Denver, Colorado
Oct. 12, 1936

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MEMORANDUM TO CHIEF DESIGNING ENGINEER

Translation from French

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INTRODUCTION

To solve by computation the concrete problems which are presented in the domain of hydraulics, the engineer has available two distinct groups of theories. The first of these is hydrodynamics, which seeks to determine the mechanism of movement of the water to the last detail by following each liquid particle. The second is hydraulics. Hydraulics, in order to simplify the difficulty of the problem, considers only the motion of the mass as a whole without troubling about the trajectories of the different particles of water.

Hydrodynamics requires the solution of a system of partial differential equations, the equations of Navier-Stokes, the integrations of which offer enormous difficulties. It is only in occasional special cases in which the symmetry of flow causes a part of the terms of these equations to vanish that it has been possible to integrate them. Up to the present no integrals of the complete equations of Navier-Stokes have been found. It has been attempted, as proposed by Oseen, to use these equations by omitting the inertia terms, which allows them to be integrated in a certain number of special cases, but the comparison of the solutions thus obtained with the results of experiments has not, in general, been satisfactory. Moreover, the equations of Navier-Stokes themselves give results consistent with experiments only when the velocities are very low, that is to say, under the conditions of viscous flow.

As an example, the equations in question have been integrated in the case of flow in a circular pipe. The distribution of velocities and pressures thus obtained corresponds almost exactly to the results of experiment, but only for very low velocities. When, however, for a given diameter of pipe and a given liquid, the velocity exceeds a certain value, streamline flow disappears and gives place to turbulent motion, characterized by more or less agitation of the liquid particles. For this motion the velocities and pressures differ entirely from those obtained by computation.

Several attempts have been made to introduce turbulence into the equations for the motion of liquids. One of the most recent is that of M. Brillouin. But as the equations which this scientist has set up are still more complicated than those of Navier-Stokes, it is apparent that for concrete cases, up to the present time, the problem has not been solved.

In almost all the problems which require solution the engineer must deal with turbulent flow. It is evident, then, that the equations of M. Brillouin, much less those of Navier-Stokes or of Oseen, will not give him what he requires; namely, rapid solutions which correspond to the facts of natural phenomena.

Recently, hydrodynamics has made remarkable progres due to a new theory called that of the boundary layer. This theory is due principally to the work of Prandtl, Blasius, von Karman, and others. It can be summarized as follows:

In a mass of fluid in motion, the internal friction is negligible. It is important only in the immediate vicinity of the sides in contact with the fluid. The very thin zone over which friction acts, caused principally by the fact that at the sides themselves the velocity must be zero, is called the boundary layer. One may then apply to the mass of the fluid all the equations and properties of the theory of perfect fluids, the equations of Euler and of Lagrange, the theory of turbulence of Helmholz, Lord Kelvin, etc. In the boundary layer, on the contrary, it is necessary to take account of friction. The boundary layer theory has, for a principal field of application, turbulent motion, to which experiments have shown that its hypotheses apply. Although its use has already permitted the quantitative solution of some problems, for example, that of the distribution of velocities in a circular pipe with a turbulent flow, it is nevertheless true that in general it gives only qualitative relations between cause and effect.

The engineer can use profitably the theory of the boundary layer. It gives him the explanation of a certain number of phenomena which he encounters - causes of the formation of eddies, for example - and it also enables him to predict the existence of certain properties of flowing streams. But this, at present, is its limit as well as that of the science of hydrodynamics.

Now what the practicing engineer needs above all is a quantitative, rapid solution of the problems with which he is confronted. This explains the great development of hydraulics, a development which has been made almost completely independent of that of hydrodynamics. As already stated, hydraulics considers in general only the motion of the mass as a whole without troubling about trajectories of the individual liquid particles. By making certain simplified hypotheses about the motions of these individual liquid particles, it is possible to establish, rapidly and easily, simple quantitative relations between the various factors which interest the engineer. Since these relations, the results of hypotheses which are often rather crude, are not always in exact accord with the facts, it is necessary to bring about agreement by the use of experimentally determined coefficients.

It is obvious, then, that experiment forms a fundamental part of hydraulics. Up to the end of the 19th century the engineers' experimental field consisted almost exclusively of observations of results from works which they had constructed, but since the beginning of the 20th century hydraulic laboratories for the study of hydraulic structures by means of reduced models have been built in various countries.

Theoretical and experimental research has permitted the classification of stream flow into two large categories; one in which similitude is possible, and one in which it is not. The first category includes flow in those works where the destruction of energy per unit of length is large; the second, flow in canals and conduits of considerable length where the loss of energy is small.

The principal objective of hydraulic laboratories is to solve by the aid of reduced models actual problems; but besides this they also endeavor to advance the science of hydraulics by attempting to obtain results which are as general as possible. Since it is difficult to derive general laws from purely experimental results, the laboratories try, when they have the time and the means, to establish equations by an application of the known laws of applied mechanics and hydraulics which will be applicable to the phenomena observed.

Which are the most useful theorems of applied mechanics for this purpose? Since hydraulics treats in general of mass movement, the general theorems concerning the motion of a mass of material points evidently will not be most applicable. Of especial importance among these are the theorems of the conservation of momentum and those of energy.

It is evident, then, that in the present state of science the easiest and most useful plan which the engineer can follow for the solution of practical problems is that of using the science of hydraulics combining the general theorems of applied mechanics with experiments made either in nature or in the laboratories, all in the field of mass movement.

The hydraulic research laboratory attached to the Federal Polytechnic School, which has been described elsewhere in detail, has endeavored, from the beginning of its existence, to follow this plan. This paper, which is an account of research carried on in that institute, shows how it has been possible, by the aid of the theorem of projection of momentum and tests on reduced models, to establish general formulae concerning the stable flow of liquids.

The first chapter, purely theoretical, is devoted to the derivation of these formulae, of which one of the principal advantages is to permit the making of a complete and easy analysis of the different causes of variations in level, pressure, and loss of energy in flowing streams. These equations show that the loss of energy is due to the three following causes:

First: The development of turbulence at the contact with the sides, which depends primarily on their roughness.

Second: The variation, along the stream axis, of the velocity distribution in the cross section.

Third: A variation in discharge, due to lateral inflow or outflow.

The first of these has already been frequently studied. For this reason, it will be discussed only in its application to the present paper.

The whole second chapter is given to the discussion of the results of model tests on the tailrace canal of a low-head power plant, in which the second cause plays a fundamental part.

Chapter III is devoted to variable discharge; that is, to the third cause of loss of energy.

Finally, chapter IV is reserved for conclusions.

The author desires to express his sincere thanks to Prof. E. Meyer Peter, director of the Hydraulic Research Laboratory, under whose direction the studies discussed in this paper were made.

CHAPTER I

STABLE FLOW IN CHANNELS AND CONDUITS WITH RECTILINEAR AXES, BUT OF ANY CROSS SECTION

Section 1. Definition of Stream Flow

This chapter will deal with stable flow in open channels and closed conduits whose axes are straight or only slightly curved. The cross section may be of any form, constant or not. It will be assumed that the surface elements of the sides and bottom make only small angles with the axis. The streams thus defined will then have, at different points in the same section, local mean velocities almost parallel to each other, and the trajectories of these velocities will be practically parallel to the axis.

For open channels, it will be assumed that the slope of the bottom is relatively small. Vertical and normal sections will then be practically identical.

The motion of the water which has just been defined is usually known by the name "courant liquide."* It is easily shown

*Translator's note: No exactly equivalent term is known in English. It has usually been translated "flowing stream", or an equivalent expression, when it occurs in the text.

that such a stream possesses the following properties:* The pressure in a given section, normal to the trajectories, varies according to the hydrostatic law, and, in consequence, if there is a free transverse surface, it is horizontal. A "courant liquide" can be either of constant or variable discharge along its length. It will be assumed that the variation in discharge, when it exists, is small and does not sensibly change the properties just given.

*See, for example: A. Flamant, Hydraulique, p 38 et seq.

Section 2. Tangential Force of a Flowing Stream at its Contact with the Wetted Perimeter

Consider a flowing stream (courant liquide) of constant discharge in a channel or conduit of great length. Over a certain part of this length flow will be uniform; that is, its characteristics (depth, velocity distribution, turbulence, etc.) will be the same, regardless of the section chosen.

During such flow the loss of energy will be constant per unit of length. The direct cause of this loss is the formation of eddies in the vicinity of the side walls which creates in the whole mass a definite turbulence. The marked internal friction, which always accompanies turbulence, transforms the mechanical energy of the water into thermal energy. As the formation of the eddies evidently depends on the roughness of the side walls, the loss of energy will be a function of this roughness, assuming other factors.

There are numerous formulae which give, for uniform flow, a relation between the constant slope J of the energy line, the mean velocity, U, the hydraulic radius, R, and a coefficient depending on the roughness of the side walls. For convenience, we shall use here Strickler's* formula, discovery of which is due to Gauckler**:

$$(1) \qquad U = k\, R^{2/3}\, J^{1/2},$$

*Strickler: Mitteilung des Amts fur Wasserwirtschaft, No. 16, Berne 1923.

**Gauckler: Annales des Ponts et Chaussées, Tome 15, 1868, p. 229.

in which k represents a coefficient depending solely on the roughness of the perimeter. However, any other formula could be used in developing this chapter.

Having assumed a relation between U, R, and J, the tangential force of the water in contact with the sides can be computed. Consider two adjacent sections of surface S at a distance dx from each other. The loss of energy between these two sections will be

$\frac{U^2}{k^2 R^{4/3}}\, dx.$* Since the motion is uniform, the corresponding change

*In this paper the expression "loss of energy" is used in the sense usual among engineers; that is, it is the loss of mechanical energy, between two cross sections, of any mass of water forming a part of the stream when it passes from one section to the other, divided by the weight of the mass. This loss has the dimension of length.

in pressure will be equal to that loss. The resultant of the pressures acting on the two cross sections will then be

$$\frac{U^2}{k^2 R^{4/3}}\, \Pi\, S\, dx,$$

where II represents the unit weight of water. This force is parallel to the current and in the same direction. Moreover, the mass of water contained between the two sections is not accelerating. The resultant, dF, of the tangential force, acting on this mass at the contact with the sides, is then equal, and opposite in direction, to the difference in pressure just indicated. Hence, Q being the discharge,

$$(2) \qquad dF = \frac{-U^2}{k^2 R^{4/3}} \, II \, S \, dx = \frac{-II \, Q^2}{k^2 S R^{4/3}} \, dx.$$

This is the expression for dF in the case of uniform motion. When the motion is otherwise, the loss of energy may have other causes than the formation of eddies in the vicinity of the sides. It will be shown in section 6 that a variation in discharge, dQ, or even a variation in the velocity diagram between two sections, will cause energy losses. The total loss can then no longer be expressed by Strickler's formula and, moreover, the variation in pressure is not equal to that loss. In the general case, therefore, it must be assumed that formula (2) gives only the tangential force acting at the contact with the sides. To elucidate this point, a series of experiments was performed, which will be discussed in section 4 of chapter III. These lead to the conclusion that formula (2) can be used, with sufficiently close approximation for technical applications, whatever may be the different causes of the loss of energy.

Section 3. Momentum and Kinetic Energy at a Cross Section

Consider any flowing stream, referred to a Cartesian system defined as follows:

If it is an open channel (fig. 1a), the X axis is chosen horizontal, and in such a way that it coincides with the average of the projections of the stream lines on a horizontal plane passing through the origin, the Z axis vertical and the Y axis perpendicular to the other two. If it is a closed conduit (fig. 1b), the X axis is chosen to coincide with the axis of the conduit; the Y and Z axes, perpendicular to each other, lie in a plane normal to the X axis.

A cross section will be, by definition, the figure formed by the intersection of the stream with a plane normal to the X axis. Each section will be denoted by its abscissa X (fig. 2). Let V be the velocity at a point A of a given cross section S, and u be the projection of this velocity on the X axis.

The projection ϕ_1, on the X axis, of the momentum per unit of time across the surface S, has the value:

$$(3) \qquad \Phi_i = \iint_{(S)} \frac{II}{g} u^2 \, d\sigma,$$

$d\sigma$ being an element of the surface containing A,

g the acceleration of gravity.

Again, let U be the mean value of the components u at section S:

$$(4) \qquad U = \frac{\iint_{(S)} u \, d\sigma}{S}.$$

We can always put

$$(5) \qquad \Phi_i = \alpha \frac{II}{g} U^2 S = \alpha \frac{II}{g} U Q,$$

in which α is a coefficient equal to or greater than unity. Eliminating Φ_i between equations (3) and (5) we obtain:

$$(6) \qquad \alpha = \frac{\iint_{(S)} u^2 \, d\sigma}{U^2 S}.$$

The projection Φ_t on the X axis of the kinetic energy per unit of time across the section S has the value:

$$(7) \qquad \Phi_t = \iint_{(S)} \frac{1}{2} \frac{II}{g} u^3 \, d\sigma.$$

We can always put

$$(8) \qquad \Phi_t = \frac{1}{2} \alpha \frac{II}{g} U^3 S,$$

in which α' is a new coefficient equal to or greater than unity. Equating the two expressions for ϕ given above,

$$(9) \qquad \alpha' = \frac{\iint_{(S)} u^3 \, d\sigma}{U^3 S} .$$

To simplify nomenclature we will call

α the coefficient of momentum at section S,

α' the coefficient of kinetic energy at that section.

These coefficients depend solely on the distribution of the components u in the section. If this distribution is known, it is easy to compute them:

analytically, if u is a known algebraic function of Y and Z,

graphically, if u is given for several points of the section.

Since α' varies as the cube of the velocity, while α varies only as its square, it is immediately evident that α' is always greater than α.

When the motion is uniform, the coefficients α and α' are generally in the neighborhood of unity. Bazin,* for example, found $\alpha' = 1.038$ for a canal of very smooth planks, and $\alpha' = 1.122$

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*Bazin: Recherches expérimentales sur l'écoulement de l'eau dans les canaux découverts.

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when the planks were covered with strips placed transverse to the current, forming very rough sides.

In nonuniform motion, α may reach values of the order of 2. It will be seen, for example, in chapter II, that at the draft tube outlets of a Kaplan turbine, the following experimental values were found:

$$\alpha = 1.23, \qquad \alpha' = 1.74,$$

but this is an exceptional case for a "courant liquide." In the majority of practical cases, α usually lies between 1 and 1.2 and α' between 1 and 1.6.

Under these conditions, if we place

$$(10) \qquad \alpha = 1 + \eta,$$

where η is a new coefficient, it is easily shown by a simple calculation that we must have

$$(11) \qquad \alpha' = 1 + 3\eta,$$

with an approximation which becomes closer as η diminishes, that is for values of α in the neighborhood of one.*

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*Flamant: Hydraulique, p. 36 et 37.

The degree of approximation of this formula depends but little on the value of η. The example given above ($\alpha = 1.23$, $\alpha' = 1.74$) shows, in fact, that even for $\eta = 0.23$ it gives good results ($3\eta = 0.69$ instead of 0.74). The exactness of formula (11) for relatively large values of η evidently depends on the nature of the velocity distribution in the section S, as is shown by the two following examples, obtained by the calculus.

(1) Parabolic distribution of velocities in a circular pipe. Solving formulae (6) and (9) by the calculus one finds:

$$\alpha = 1.333.... \qquad \alpha' = 2,000,$$

$$\text{whence:} \quad \eta = 0.333 \quad \text{and } 1 + 3\eta = 2,000.$$

Formula (11) gives the exact value of α'.

(2) Parabolic distribution of velocities in a stream between two parallel sides (two dimensional motion). The calculus gives:

$$\alpha = 1,200 \qquad \alpha' = 1.543,$$

$$\text{whence:} \quad \eta = 0.200 \quad \text{and } 1 + 3\eta = 1,600.$$

The error in the value of α' given by formula (11) is here

$$\frac{1.600 - 1.543}{1.543} = 0.037, \text{ or } 3.7 \text{ percent.}$$

This is evidently an extreme value.

In practical cases, the following degrees of precision may be expected for values of α' obtained by use of formula (11):

0.1 to 0.2% for $1 < \alpha < 1.1$; that is, for $1 < \alpha' < 1.3$,

1.0 to 2.0% for $1.1 < \alpha < 1.2$; that is, for $1.3 < \alpha' < 1.6$.

In what follows in this chapter, η will be considered sufficiently small, with respect to unity, to be considered as an infinitesimal of the first order. Practically, this will always be the case when η lies between zero and 0.1, that is, for values of $1.0 < \alpha < 1.1$.

Section 4. The Theorem of Projection of Momentum Applied to Hydraulics

Consider any liquid in motion (fig. 3). Let S be a closed surface, situated entirely within the liquid. It encloses a known mass, M. Assume that the points of the surface S are displaced with the molecules of the liquid. In this case the mass M will change position and shape with time always, however, being composed of the same part of the liquid. The mass M may then be considered as composed of a perfectly definite assembly of material points, and the theorem of projection of momenta applied may be stated as follows:*

*The demonstration of this theorem will be found in "Traité de Mécannique rationelle" d'Appel, Tome II, pages 18 et seq.

The derivative, with respect to time, of the sum of the projections of momenta of all the elements of a given mass of liquid, on any fixed axis, is equal to the sum of the projections of the exterior forces on that axis.

Section 5. General Equations of Stream Flow (Stable Motion)

Assume any stream whatever which satisfies the conditions set forth in section 1 of this chapter. This may be either in an open channel (fig. 4) or in a closed conduit under pressure (fig. 5). Assume, also, that it may receive an increment of flow, or suffer a decrement, so that the discharge will be variable along the channel.

Consider, at time t, the mass of water ABCD contained between two planes perpendicular to the axis of abscissae at X and X + dx.

At x let S = area of cross section,
σ = element of area,
Q = discharge,
u = component of velocity parallel to the X-axis at any point in the section,
U = average value of this component,
α = coefficient of momentum,

α' = coefficient of kinetic energy,

z = elevation of the water surface, or of the hydraulic gradient,

$z + \alpha' \frac{U^2}{2g}$ = elevation of the energy gradient.*

*This elevation represents the total energy, at the section considered, of a mass of water forming a part of the stream, divided by its weight.

Corresponding quantities at the section at $x + dx$ are:

$$S + dS,\ d\sigma,\ Q + dQ,\ u + du,\ U + dU,\ \alpha + d\alpha,\ \alpha' + d\alpha',$$

$$z + dz,\ z + dz + (\alpha' + d\alpha')\ \frac{(U + dU)^2}{2g}.$$

Here dQ represents the change in discharge (positive or negative) over the length dx. Assume that dQ enters the length dx with a velocity V^*, having a component u^* parallel to the x axis. Also

*dH_e is by definition the change in height of the energy line between the two sections. In the general case, where this discharge is variable along the stream, it may be either positive or negative. For constant discharge it is negative and represents a loss of energy. This term was defined by note 4, sec. 2.

denote by dH_e the change in energy between the two sections considered,* by $d\alpha'$ an element of the surface dS, the latter being the projection of the lateral surfaces of the mass considered on a plane perpendicular to X, and by $dp = \Pi \cdot dz$, the increase in pressure between any point on the section at X and a point at the same depth on the section at X + dx.

Let us now apply the theorem of the projection of momentum to a mass of water, composed, at time t, of the two following parts:

(1) the mass included in the space ABCD,

(2) A mass $\frac{w}{g}$ dQdt which is the increment of inflow over the length dx during the time dt.

During this time interval, the mass $\frac{w}{g}$ dQ dt mingles with the stream so that at the time t + dt the total volume (1) plus (2) occupies the space A'B'C'D' (figs. 4 and 5).

At the time t, the sum of the projections on the X-axis of the momenta of the mass (1) + (2) has a known value which may be called $J_o(t)$.

At the time t + dt, the sum of the projections of the momenta of this same mass (1) + (2), which occupies at that instant the space A'B'C'D' is

$$J_o(t + dt) = J_o(t) + \int\int_{(S+dS)} \frac{w}{g}(u + du)^2 \, d\sigma \, dt - \int\int_{(S)} \frac{w}{g} u^2 \, d\sigma \, dt - \frac{w}{g} u^* \, dQ \, dt.$$

In this equation the term $\frac{w}{g} u^* dQdt$ represents the inclusion of the mass (2) defined above in the space A'B'C'D'.

The derivative, with respect to time, of the sum of the momenta will then be:

$$\frac{dJ_o}{dt} = \frac{J_o(t + dt) - J_o(t)}{dt} = \int\int_{(S + dS)} \frac{w}{g} (u + du)^2 \, d\sigma - \int\int_{(S)} \frac{w}{g} u^2 \, d\sigma - \frac{w}{g} u^* dQ,$$

or, introducing the coefficients of momentum and replacing the product US by Q:

$$\frac{dJ_o}{dt} = \frac{w}{g} \left[(\alpha + d\alpha)(U + dU)(Q + dQ) - UQd\alpha - u^* \, dQ \right].$$

Developing this expression and neglecting infinitesimals of the 2nd and 3rd orders, it becomes:

$$(12) \quad \frac{dJ_o}{dt} = \frac{w}{g} \left[(\alpha U - u^*) \, dQ + SU(\alpha \, dU + Ud\alpha) \right].$$

Now, the sum F of the projections on the X-axis of the exterior forces acting on the mass (1) + (2) under consideration, is expressed by:

$$F = -\int\!\!\int_{(S)} dp\,d\sigma - 1/2 \int\!\!\int_{(dS)} dp\,d\sigma' - \frac{wQ^2}{k^2 S R^{4/3}}\,dx = -dp(S + 1/2\ dS)$$

$$- \frac{wQ^2}{k^2 S R^{4/3}}\,dx,$$

Neglecting infinitesimals of the second order this becomes:

$$(13) \qquad F = -Sdp - \frac{wQ^2}{k^2 S R^{4/3}}\,dx.$$

Equating the value of $\frac{dJ_o}{dt}$ given by equation (12) to the value of F in equation (13) we have:

$$-Sdp - \frac{wQ^2}{k^2 S R^{4/3}}\,dx = \frac{w}{g}\left[(\alpha U - u^*)dQ + SU\,(\alpha\,dU + Ud\alpha)\right].$$

This last equation permits us to calculate $dz = \frac{dp}{w}$; recalling that Q = US, we obtain:

$$(14) \qquad dz = \frac{dp}{w} = \frac{U^2}{g}\left[-\frac{g}{k^2 R^{4/3}}\,dx - d\alpha - (\alpha - \frac{u^*}{U})\,\frac{dQ}{Q} - \alpha\,\frac{dU}{U}\right].$$

This is the first general equation for stream flow. We can obtain a second by computing the change of energy dH_e. A glance at figures (4) and (5) shows that:

$$\alpha'\,\frac{U^2}{2g} + dH_e = dz + (\alpha' + d\alpha')\,\frac{(U + dU)^2}{2g},$$

from which, neglecting infinitesimals of the 2nd and 3rd orders:

$$dH_e = \frac{\alpha' U}{g}\,dU + \frac{U^2}{2g}\,d\alpha' + dz.$$

Substituting in this equation the value of dz from (14) there results, after some simplifications:

$$dH_e = \frac{U^2}{2g}\left[\frac{-2g}{k^2R^{4/3}}\,dx - 2d\alpha + d\alpha' - 2\left(\alpha - \frac{u^*}{U}\right)\frac{dQ}{Q} + 2(\alpha' - \alpha)\frac{dU}{U}\right].$$

From equations (10) and (11) we have the relations:

$$\alpha' - \alpha = 2\eta,$$

$$d\alpha = d\eta, \quad d\alpha' = 3\,d\eta,$$

$$d\alpha' = 3\,d\alpha,$$

$$-2\,d\alpha + d\alpha' = +\,d\alpha$$

Substituting the values for $-d\alpha + d\alpha'$ and $\alpha' - \alpha$ in the expression for dH_e, we obtain the following equation:

$$dH_e = \frac{U^2}{g}\left[-\frac{g}{k^2R^{4/3}}\,dx + \frac{1}{2}\,d\alpha - \left(\alpha - \frac{u^*}{U}\right)\frac{dQ}{Q} + 2\eta\frac{dU}{U}\right].$$

Now, if we consider η as an infinitesimal of the first order, as noted at the end of the preceeding paragraph, the term $2\eta\frac{dU}{U}$ can be neglected as an infinitesimal of the second order (assuming evidently that U is not itself an infinitesimal), and we have the second general equation of stream flow:

$$(15) \qquad dH_e = \frac{U^2}{g}\left[-\frac{g}{k^2R^{4/3}}\,dx + 1/2\;d\alpha - \left(\alpha - \frac{u^*}{U}\right)\frac{dQ}{Q}\right]$$

Section 6. Discussion of the General Equations of Stream Flow (14) and (15)

Equation (14) gives the change $dz = \frac{dp}{w}$ of pressure between the two sections at X and X + dx. In the case of an open channel, dz is nothing more nor less than the change in elevation of the water surface. Equation (15) gives the change in energy, dH_e, between the two sections considered.

The two formulae have exactly the same form; that is, dz and dH_e are each proportional to the square of the mean velocity U and to a factor containing several terms, four for dz and three for dH_e. These terms represent the different elements that go to make up the total of the quantities dz and dH_e. The three terms in the bracket in (15) are nearly identical with those in (14); the latter equation, however, contains an additional term peculiar to itself.

Let us examine the elements represented by the different terms in the parenthesis.

(1) The roughness of the surfaces in contact with the fluid gives the term $- \frac{g}{k^2 R^{4/3}} dx$ in formula (14) and the same term in formula (15). This roughness, then, always causes a decrease in pressure and a loss of energy which are equal to each other.

(2) The variation in distribution of velocities, U, in the section, in passing from the section at x to that at x + dx, results in the term $-d\alpha$ in the expression for dz and $+1/2\, d\alpha$ in that for dH_e. The momentum coefficient α depends, in fact, directly on the velocity distribution. A variation of this distribution influences the values of dz and dH_e only if α varies by a quantity $d\alpha$. It is evident that a decrease in the coefficient α ($d\alpha$ negative) results in a rise in the pressure line, or water surface (dz positive) and a loss of energy (dH_e negative), the latter being, in absolute value, one half of the rise in dz. Conversely, an increase in α ($d\alpha$ positive) results in a drop in the pressure line, or water surface (dz negative), and an increase in energy (dH_e positive).

If the discharge is constant ($dQ = 0$) this increase in energy must necessarily be smaller, in absolute value, than the loss in energy due to the roughness of the surface, for according to the second principle of thermodynamics, the total mechanical energy can only decrease.

(3) The increase or decrease in discharge dQ in the length dx gives the term $-(\alpha - \frac{u^*}{U}) \frac{dQ}{Q}$ in the formula for dz and the same term in that for dH_e. The value of dz or of dH_e corresponds to the variation in discharge dQ, being positive or negative according to the sign of dQ and the relative values of u^*, U, and α.

Consider the case when dQ is positive (discharge increasing). Two cases may be distinguished:

(1) $u^* < \alpha U$. The variation in discharge causes a drop in

the hydraulic gradient and a loss of energy. In particular, if $u^* = 0$ (the increment of discharge dQ enters the stream with a velocity perpendicular to the direction of the current) a loss of energy will always result.

(2) $u^* > \alpha U$. The variation dQ causes a rise in the hydraulic gradient and an increase in energy. This increase, moreover, is compatible with the second principle of thermodynamics, since it is caused by an addition of water from outside the main stream.

Consider, now, dQ negative (discharge decreasing). Exactly the opposite phenomena occur. In particular, if $u^* = \alpha U$ (hypothesis made, for example, in the case of an open channel with a side spillway parallel to the direction of flow) there will be no change in energy.

(4) The variation in mean velocity, U, between the sections at x and at x + dx causes a variation of pressure, dz, expressed by the fourth term $-\alpha \frac{dU}{U}$ in the bracket in formula (14). However, the variation in mean velocity causes no change in energy, since this term is absent in formula (15). If the mean velocity increases ($dU > 0$), the pressure drops ($dz < 0$), and if it decreases ($dU < 0$), the pressure rises ($dz > 0$). It is to be noted that in a concrete case dU can always be expressed as a function of the variation in discharge, dQ, and the variation of the area of the section dS. In the case of a closed conduit, dS is a function of the dimensions of the channel only, while in an open channel it depends on the variation in depth, dz, as well. This is, in fact, the only difference between the two classes of channels.

In computing surface curves in open channels, the engineer ordinarily considers only the first and last factors given above (roughness of the sides and variation in mean velocity). This assumes a constant velocity distribution ($d\alpha = 0$) and a constant discharge ($dQ = 0$). Equation (14) then becomes:

$$-dz = \frac{U^2}{k^2 R^{4/3}} dx + \alpha\, d\left(\frac{U^2}{2g}\right),$$

or, designating by the subscript 1 the properties of section x, by subscript 2 the properties of section x + dx, and by subscript m the average values; and replacing differentials by finite differences:

$$(16) \qquad -\Delta z = \frac{U_m^2}{k^2 R_m^{4/3}} \Delta x + \alpha \frac{U_2^2 - U_1^2}{2g} .$$

This equation permits calculation, step by step, of any number of points on the surface curve, beginning at a section where the depth is known.* In the case of streaming flow, this section

*Note that in formula (16) α is the coefficient of momentum. In deriving this formula directly from Bernoulli's theorem, as is usually done, it is found that the coefficient by which $\frac{U_2^2 - U_1^2}{2g}$ must be multiplied must be equal to α' (coefficient of kinetic energy). This difference, which arises from the fact that equation (16) has been obtained by another than the classical method is, however, infinitesimal. It has been assumed (par. 3) that η is an infinitesimal, hence $\alpha' - \alpha + 2\eta$ is also an infinitesimal.

is at the downstream end of a reach; the calculation proceeds upstream. In the case of shooting flow, it is necessary to proceed in the opposite direction, for in this case the section of known depth is at the upstream end of the reach.

The above equation can also be used for the determination of the hydraulic gradient in closed conduits of variable section. When the section is constant, it reduces to

$$-\Delta z = \frac{U_m^2}{k_2 R_m^{4/3}} \Delta x.$$

In general, the results given by formula (16) are satisfactory, but there are numerous cases in which the value of dz depends principally on the variation in velocity distribution $d\alpha$ and the discharge dQ.

It may be admitted at once that the introduction of a variation in the coefficient α in any calculation is impracticable since, in general, nothing is known as to its variation. Moreover, the flow of liquids takes place in fashions so diverse, and even so contrary to what might be expected, that it would be useless to attempt a hypothesis on the subject. The introduction of the value of $d\alpha$ into calculations must be confined to those cases in which it can actually be measured. In chapter II is given an example of its measurement by means of the Pitot tube. Although it is difficult to obtain quantitative results from equations (14) and (15) when $d\alpha$ is to be considered, still these equations will give useful qualitative indications of the variation in velocity distribution. We will show, for example, in the following paragraph that by means of equation (15) it may be proved that flow at constant discharge is stable. Moreover, the presence of the term $d\alpha$ in this equation will sometimes account for anomalies observed in the flow of a stream, by showing that a part of the energy loss may be attributed to redistribution of velocities.

On the other hand, the influence of a variation in discharge can, in general, be easily treated mathematically since in a majority of practical cases this variation is a known quantity. In chapter III will be given several examples of varying discharge, and it will be shown that the results arrived at from use of the general equations for stream flow have been confirmed by experiments on reduced models.

Remark. Equations (14) and (15) can be still more generalized to cover the case in which the increment of discharge, dQ, enters the stream with a nonuniform velocity distribution. It is only necessary to replace u^* by $\alpha^* U^*$ where α^* represents the momentum coefficient of the increment dQ, and U^* the component, parallel to the axis, of the mean velocity of the increment.

Section 7. Stability of Velocity Distribution in a Stream of Constant Discharge

Consider uniform flow (courant stationnaire) in an open canal or a closed conduit (fig. 6). For a short distance, in terms of the hydraulic radius, the influence of the roughness of the sides can be neglected. Formula (15) then becomes:

$$(15a) \qquad dH_e = + \frac{U^2}{2g} d\alpha .$$

Suppose that at a section, A, there exists a disturbance such as might be caused, for example, by a bar across the stream. This disturbance will result in quite an uneven velocity distribution at section II, a short distance downstream from A. It is desired to know whether this disturbance will have a tendency to decrease or increase in a downstream direction. In the first case the flow will be stable, in the second unstable. By the use of formula (15a) it can be shown that the flow is stable. For, since by hypothesis the discharge is constant, dH_e must be zero or negative, which requires that $d\alpha$ itself must be zero or negative. Under this condition, the inequality of velocity distribution downstream of section II must diminish ($d\alpha < 0$) or at least remain constant ($d\alpha = 0$). The velocity diagrams will then have a tendency toward uniformity as shown in figure 6 (sections III and IV). This stability of flow is fully confirmed by experience. It is known, for example, that the unequal velocities caused in a conduit by a sudden enlargement of cross section, a bend, a valve, etc., always disappear at a certain distance from the cause of the disturbance.

CHAPTER II

STUDY OF THE TAILRACE CHANNEL OF A LOW-HEAD POWER PLANT

Section 1. General

The purpose of the draft tubes of low-pressure turbines is for transforming the kinetic energy of the water at the exit from the runner into potential energy.

In installations of high specific speed, this kinetic energy is quite large. Its recovery - that is, its transformation into potential energy - must be effected very carefully, for the energy recovered is used directly by the turbines. An ideal draft tube would be one in which this recovery is effected without loss of energy and in which the kinetic energy at the discharge and of the channel would be reduced to a minimum value of $\frac{U^2}{2g}$, corresponding to a uniform velocity distribution across the section. In practice this ideal is never attained. The remaining kinetic energy at the outlet of the draft tube has a value $\alpha\frac{U^2}{2g}$ very much larger than $\frac{U^2}{2g}$ since α has a value of at least 1.5. Besides, the junction of the draft tubes with the tailrace is always characterized, for construction reasons, by a sudden enlargement of wetted sections; and for this reason the velocity distribution at the upper end of the tailrace is always very irregular. However, due to the stability of flow mentioned at the end of the preceeding chapter, this inequality of velocities diminishes rapidly, and at a relatively short distance from the entrance there exists in the canal a normal distribution of velocities with a coefficient of kinetic energy α, in the neighborhood of 1.

Evidently the flow in the tailrace channel of a low-head power plant presents a typical case of marked variation in velocity distribution. At the entrance section, in consequence of the unequal velocity distribution, the kinetic energy is relatively great, while at the exit section the kinetic energy is, in general, about equal to $\frac{U^2}{2g}$. In a way, the canal continues the work, begun by the draft tubes, of recovering as large a part as possible of the kinetic energy and transforming it into potential energy. No difference exists, in principle, between the draft tube and the canal; the only distinction is that the former is under pressure and the latter is not.

The tailrace channel offers the same problem as the draft tube, that of recovering kinetic energy with a minimum of loss. Evidently the recovery in the channel is always very much smaller than in the draft tube. While in the latter it may be two to three meters, in the former it may be in a well-designed channel from one to two decimeters. Since in very low-head plants one decimeter represents, nevertheless, an appreciable percentage of the total net head, it is generally worth the trouble, especially in important installations, to study as carefully as possible the design of the tailrace channel. For this reason the S. A. Rheinkraftwerk Albbruck-Dogern retained the hydraulic research laboratory of Zurich to make tests on reduced scale models for the purpose of obtaining the best design for the tailrace channel for the Dogern hydroelectric plant on the Rhine, now under construction. These tests were made under the direction of Dr. H. E. Gruner, consulting engineer of Basel, who initiated the project and directs its execution.

Section 2. Description of the Tailrace Channel Studied by Reduced Scale Models and of the Test Installation

The hydroelectric installation in question includes a dam with gates on the Rhine, an intake, a power canal about 3.5 km long, a power plant with 3 Kaplan turbines with a normal discharge of 250 cu. m. per sec. each, under a head varying between 7.5 and 11.5 meters, and a tailrace channel returning the water to the river.

This tailrace channel is 150 meters long. In plan, its axis is the arc of a circle of 420 meters radius. Its transverse section is a rectangle with a width of 79 meters at the power-house end and 100 meters at the other end. The determination of the longitudinal profile was the object of the present study.

In order to study different designs of the tailrace channel, the laboratory built a model to a scale of 1:34 of part of the power canal, the power plant with its 3 Kaplan turbines, and the tailrace channel (figs. 7, 8, 9). Figure 9 shows plan and elevation of the test installation. From left to right are the supply pipe with regulating valve, a measuring weir for determining the discharge, a stilling pool with baffles and floating planks, the model proper, and a tank representing the Rhine, with a gate for regulating the height of tail water in the river.

A coordinatograph, with a large carriage rolling on rails parallel to the axis of the canal and with a small carriage moving at right angles thereto, was used to measure the water surface and to hold the Pitot tube for measuring velocities in the channel (fig. 9). Either a point gage or a Pitot tube could be fastened on the small carriage (fig. 12).

It may be added that the installation permitted exact determination of the power developed by the turbines by means of an electric brake, the torque being measured by a balance and the number of revolutions by tachometers (fig. 7).

Section 3. The Tests and Their Results

The elevation, figure 9, shows the longitudinal profile of different designs of the tailrace channel. Designs I-A, I-B, and I-C refer to draft tube I and design II-A to draft tube II.

First method.--The first method of measurement consisted of determining the over-all efficiency η_{o3} between sections 0 and 3 (see elevation, fig. 9) for each of the four designs of the tailrace channel shown. The net power was obtained by measuring simultaneously the torque with the balance and the number of revolutions with the tachometers. The gross power was obtained from the discharge over the weir, and the fall between sections 0 and 3 measured by means of the coordinatograph. The quotient of the two gave the efficiency η_{o3}.

Measurements were made for discharges corresponding to 500, 755, and 900 cu. m. per sec. (total discharge of the three turbines). For each discharge for each longitudinal profile the efficiency η_{o3} was obtained as a function of the water surface elevation at section 3. Only the results for a discharge of 755 cu. m. are shown. Figure 10 shows curves of values of η_{o3}. Figure 11 shows, for example, the curve IC with the plotted points.

Comparison of the curves of figure 10 shows that designs I-A, I-B, and I-C give efficiencies averaging about 0.7 percent greater than design II-A. This permitted elimination of the latter design. The efficiencies of design I differed only slightly from each other. However, that of design I-A is not quite as good as those of designs I-B and I-C.

The following method was used for studying the action of the tailrace channel itself for designs I-A and I-C.

Second method.--This consisted of determining the loss of energy in the tailrace channel between sections 1 and 3 (figs. 9 and 13). The position of the energy line, for any profile, was obtained by plotting points above the water surface at a height corresponding to the kinetic energy $\alpha' \frac{U^2}{2g}$.

To obtain the loss of energy between sections 1 and 3 it was necessary:

(1) To measure the water surface in the model at those two sections. This was done with the coordinatograph.

(2) To determine with the Pitot tube the velocity distributions at the two sections from which was obtained the coefficient of kinetic energy α' (chapter I, section 3). The velocity distribution at section 1 is shown in figure 14. The value of the coefficient, α', for the draft tube section is 1.74; for the section of the canal immediately below it varies between 4.05 and 12.66, according to the depth of water in the channel. The velocity distribution at section 3 is normal ($\alpha' = 1.06$).

Figure 15 shows, for designs I-A and I-C, the measured values of Δz (difference of surface elevation between sections 1 and 3) as a function of water-surface elevation at section 3.

Figure 16 shows corresponding values of ΔH_e (loss of energy between sections 1 and 3) determined as described.

In these two figures the points for design I-A are indicated by crosses, and those for design I-C by circles. (It should be remembered that the curves shown on the figures exist only momentarily). It will be seen at once that the scattering of the points, due to accidental errors of measurement, renders impossible the deduction of any law. The principal source of error lies in the measurement of the water surface, which is a very delicate operation because of the small dimensions to be measured (to a scale of 1:34, Δz is of the order of 2 mm) and the inevitable pulsations of the flowing stream.

To avoid this negative result an artifice was used which nullifies the effect of accidental errors of measurement. This consists in applying the momentum theory, first to the mass of water between sections 1 and 2 which permits calculation of the difference of water level between these two sections Δz_1, and then to the mass between sections 2 and 3 which permits calculation of Δz_2. Details of the calculations, which are similar to those of chapter 1 (sec. 5) are not given here. They lead to a result expressed by the two following formulas (see fig. 13):

$$(17)\quad \Delta z_1 = \frac{n_1\left[\dfrac{\alpha_1}{b_1 h_1} - \dfrac{\alpha_2}{b_2(h_1 - m_1)}\right]}{2h_1 - m_1 - \dfrac{n_1\alpha_2}{b_2(h_1-m_1)^2}},\quad \Delta z_2 = \frac{n_2\left[\dfrac{\alpha_2}{b_2 h_2} - \dfrac{\alpha_3}{b_3(h_2 - m_2)}\right]}{2h_2 - m_2 - \dfrac{n_2\alpha_3}{b_3(h_2-m_2)^2}}.$$

To obtain the loss of energy between sections 1 and 3 it was necessary:

(1) To measure the water surface in the model at those two sections. This was done with the coordinatograph.

(2) To determine with the Pitot tube the velocity distributions at the two sections from which was obtained the coefficient of kinetic energy α' (chapter I, section 3). The velocity distribution at section 1 is shown in figure 14. The value of the coefficient, α', for the draft tube section is 1.74; for the section of the canal immediately below it varies between 4.05 and 12.66, according to the depth of water in the channel. The velocity distribution at section 3 is normal ($\alpha' = 1.06$).

Figure 15 shows, for designs I-A and I-C, the measured values of Δz (difference of surface elevation between sections 1 and 3) as a function of water-surface elevation at section 3.

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$$(17) \quad \Delta z_1 = \frac{n_1\left[\dfrac{\alpha_1}{b_1 h_1} - \dfrac{\alpha_2}{b_2(h_1 - m_1)}\right]}{2h_1 - m_1 - \dfrac{n_1 \alpha_2}{b_2(h_1 - m_1)^2}}, \quad \Delta z_2 = \frac{n_2\left[\dfrac{\alpha_2}{b_2 h_2} - \dfrac{\alpha_3}{b_3(h_2 - m_2)}\right]}{2h_2 - m_2 - \dfrac{n_2 \alpha_3}{b_3(h_2 - m_2)^2}}.$$

In these formulas:

$\Delta z_1, \Delta z_2$ = the differences in water level between sections 1-2 and 2-3,

h_1, h_2 = the depths at sections 1 and 2,

b_1, b_2, b_3 = the average widths of wetted sections 1, 2, and 3,

m_1, m_2 = the rise in canal bottom between sections 1-2 and 2-3,

$\alpha_1, \alpha_2, \alpha_3$ = the respective coefficients of momentum at the three sections, determined by Pitot tube measurements of the velocity distributions,

$n_1 = \dfrac{4\,Q^2}{g(b_1 + b_2)}$, $n_2 = \dfrac{4\,Q^2}{g(b_2 + b_3)}$ = functions easily computed from the discharge Q, the widths b_1, b_2, b_3, and the acceleration of gravity, g.*

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*It may be remarked that the influence of the roughness of the sides on the values of Δz is here completely negligible. Computing the amount of this influence, using a value of k = 100 for the model, a corrective term is obtained of the order of one-tenth of a millimeter in the model, or one-half centimeter in the prototype.

In the application of these formulas, use of approximate values of h_1 and h_2 is sufficient. The values of Δz thus obtained are independent of the accidental errors due principally, as has been said, to the measurements of the water surface. Each of the equations (17) is nothing but the integral of equation (14), chapter I, in which the influence of the roughness of the sides has been neglected.

Having computed Δz_1 and Δz_2, the value of Δz is obtained by adding these two quantities. The result of the calculations is shown by the two curves of figure 15. It is seen that the curves fit the plotted points very well. Knowing Δz, it is easy to compute ΔH_e, which is shown by the two curves of figure 16. These also fit very well the points plotted from direct measurement. It is evident from an examination of the latter figure that design I-A gives a loss of energy about 3 cm greater than design I-C, corresponding to a difference in over-all efficiency of about 0.4 percent. This result agrees with that given by the first method in which the difference was 0.3 percent.*

*It must be admitted that the agreement between the results of the two methods was not so close for the other discharges (500 and 900 cu. m. per sec.) in which there was a difference in the overall efficiency of the order of 0.5 percent. This is due to the fact that the second method takes account only of the phenomena occuring in the tailrace channel, while the first includes also turbines and the draft tubes. The divergence of results given by the two methods is explained by the fact that the form of the channel has an influence on the efficiency of the draft tubes, an influence taken into account by the first method but not by the second.

Design I-C gave the best efficiency for a discharge of 755 cu. m. The other tests showed that for a discharge of 500 cu. m. the efficiency of design I-C was about 0.1 percent greater than that of design I-A, while for 900 cu. m., design I-A gave an efficiency 0.2 percent better than design I-C. When these figures were applied to the annual discharge curve of the power plant it was found that the two designs had practically the same average efficiency. However, since design I-C had the advantage of much lower first cost (less excavation and lower side walls), this was the design recommended.

Figure 15 shows that for designs I-A and I-C the water surface rises between sections 1 and 3 (Δz positive). Since Δz is the change in potential energy between these sections, this increase in surface elevation represents that part of the kinetic energy at the entrance to the canal which has been transformed into potential energy; that is, the "recovery" mentioned at the beginning of the chapter. It is easy to see how this recovery increases the power delivered by the turbines. A positive value of Δz results in a water surface at section 1 lower than that at section 3, increasing the available head measured between sections 0 and 1.

Figure 17 shows, for designs I-A (solid lines) and I-C (dotted lines), the energy at entrance and discharge of the tailrace channel, the loss of energy and the recovery, Δz. This figure shows clearly the variation of energy in the channel.

Section 4. Conclusions

As a result of this study of an actual case, it was possible to select, with certainty, an economical design for the tailrace channel of a hydroelectric plant. Moreover, it is clearly shown that this channel recovers energy by transforming a part of the available kinetic energy into potential energy.

The application of the theory of conservation of momentum leads to equation (17), which is the integration of equation (14), and which permits calculation of the values of Δz and ΔH_e in the channel. It must be remarked that equation (17) could be used for this purpose only because of the availability of measurements of the velocity variation at sections 1, 2, and 3 which permitted determination of the change in the coefficient α. As noted in paragraph 6 of the preceding chapter, it is possible to introduce this change into the calculations only when it can be experimentally determined.

CHAPTER III

Translator's note: Translation of p. 36 has not been included. It doesn't seem to add much.

FIRST PART

SURFACE CURVES FOR INCREASING DISCHARGE, WITH LATERAL INFLOW PERPENDICULAR TO THE DIRECTION OF FLOW

Section 1. Calculation of the Surface Curve for Increasing Discharge in an Open Channel of Any Section

All of the first part of this chapter will be based on the following hypotheses:

(1) The discharge increases in a downstream direction according to a known law. The lateral inflow then is always positive ($dQ > 0$).

(2) The inflow is at right angles to the direction of flow in the channel ($u^* = C$).

(3) For simplification, the variation in the coefficient of momentum, $d\alpha$, will be neglected and α assumed equal to unity. Under these conditions equation (14) becomes:

$$(18) \qquad dz = -\frac{U^2}{g}\left[\frac{g}{k^2 R^{4/3}}\,dx + \frac{dQ}{Q} + \frac{dU}{U}\right].$$

This is the differential equation of the surface curve. In section 3 a case will be given in which it is possible to integrate the equation, but in most practical cases it is necessary to compute the curve by arithmetic integration using finite differences. For this purpose the equation (18) may be put in the form:

$$(19) \qquad -\triangle z = +\frac{U_m^2}{k^2 R_m^{4/3}}\,\triangle x + \frac{Q_2^2 - Q_1^2}{2g\,S_m^2} + \frac{U_2^2 - U_1^2}{2g},$$

in which (see fig. 18):

$\triangle x$ = the horizontal distance between two consecutive sections 1 and 2,

Q_1, Q_2, U_1, U_2 = the discharges and velocities at these sections,

S_m, R_m, U_m = the average values of wetted area, hydraulic radius and mean velocity between the sections,

k = the coefficient of rugosity of the channel in Strickler's (Manning's) formula,

Δz = the difference in elevation of water surface between sections 1 and 2. Note that if $Q_1 = Q_2$ corresponding to constant discharge, the formula reduces to the well-known formula (16), with $\sigma = 1$.

The application of formula (19) offers no difficulty, provided that at least one point on the surface curve is known. If L is the section at the downstream end of the reach over which the discharge varies, two cases may be distinguished.

(1) Streaming flow downstream of L. In this case the water surface at L is known from conditions existing downstream. The water-surface elevation at L furnishes a starting point for computing, by formula (19), the surface curve for the reach over which the discharge varies, working from L upstream.

(2) Shooting flow downstream of L. In this case conditions downstream of L have no effect on the water surface at L. The section controlling the depth at L will lie in that part of the channel over which the discharge varies. How can it be located? Very simply - as shown by Dr. Boss, by applying Gauss' principle of least work in such a way that the energy line will occupy the lowest possible position. The minimum height of the energy line will, in general, be found by assuming the critical depth to occur at L. The surface curve in that part of the channel over which the discharge varies may then be calculated, proceeding upstream. If at any point it is impossible to find a value of Δz which will satisfy equation (19), it is because the assumption of critical depth at L does not give the lowest position of the energy line. In this case it is necessary to assume critical depth at another point in the reach over which the discharge varies, as a starting point, and calculate the surface upstream and downstream from that point. This point must be such that equation (19) can be satisfied over the whole length of the channel. In this way the surface curve corresponding to the lowest position of the energy line can be obtained.

Section 2. Rectangular Channel of Constant Slope with Discharge Increasing Downstream in Proportion to Distance

Consider a rectangular channel of width b and constant slope i (fig. 19). Assume that over the length l between points O and L the inflow per unit of length is constant and equal to $\frac{Q_L}{l}$.

Assume that the water cannot escape to the left across section O.

Under these conditions flow in the channel will be from left to right, the discharge having a linear variation between the value zero at O and Q_L at L. It is proposed to study the form of the surface curve, O'L', of the water in the channel.

Let dimensions be referred to a rectangular system of coordinates xOz, as indicated in the figure. At section x let:

- $-m$ = the elevation of the channel bottom,
- z = elevation of water surface,
- U = the mean velocity,
- h = the depth,
- h_l = the depth at section L.

The discharge at any section, x, will be:

$$(20) \qquad Q = Q_L \frac{x}{l}.$$

Also, the following relations are evident:

$$(21) \quad Q = bhU, \qquad (22) \quad h = z + m. \qquad (23) \quad m = ix.$$

The influence of the roughness of the sides, which is generally small, is neglected. Equation (18) then may be written:

$$(24) \qquad dz = -\frac{U^2}{g}\left[\frac{dQ}{Q} + \frac{dU}{U}\right].$$

Writing equations (20) and (21) in logarithmic form and differentiating:

$$\frac{dQ}{Q} = \frac{dx}{x},$$

$$\frac{dQ}{Q} = \frac{dh}{h} + \frac{dU}{U},$$

$$\frac{dU}{U} = \frac{dQ}{Q} - \frac{dh}{h} = \frac{dx}{x} - \frac{dh}{h}.$$

Differentiating equations (22) and (23):

$$dh = dz + dm, \qquad dm = idx,$$

hence, $\quad dz = dh - idx.$

From (21) and (20):

$$U^2 = \frac{Q^2}{bh^2} = \frac{Q_L^{\,2}}{b^2 l^2} \cdot \frac{x^2}{h^2}.$$

Substituting the values of $\frac{d\lambda}{Q}$, $\frac{dU}{U}$, dz, and U^2 in equation (24), we obtain:

$$dh - idx = -\frac{Q_L^{\,2}}{gb^2 l^2}\,\frac{x^2}{h^2}\left[\frac{dx}{x} + \frac{dx}{x} - \frac{dh}{h}\right],$$

and recalling that $\frac{Q_L^{\,2}}{gb^2}$ is the cube of the critical depth h_k at L:

$$(25) \qquad h_k = \sqrt[3]{\left(\frac{Q_L}{b}\right)^2 \frac{1}{g}},$$

we finally obtain, after some simplification, the differential equation of the surface curve in terms of x and h:

$$(26) \qquad \frac{dh}{dx} = \frac{i\,l^2 h^3 - 2\,h_k^{\,3}\,x\,h}{l^2 h^3 - h_k^{\,3}\,x^2}. \qquad (o < x < l)$$

This expression is not one which can be integrated. However, it discloses the character of the surface curve.

For $x = 0$, $\frac{dh}{dx} = i$, and hence from (22) and (23) $\frac{dz}{dx} = 0$ which is to say that the tangent at O' is always horizontal.

First case.--Assume that the denominator, $l^2 h^3 - h_k^{\,3} x^2$, does not vanish for any value of x between 0 and l; that is, that $\frac{dh}{dx}$ is continuous over that interval. Under this hypothesis, this denominator must remain positive, since for $x = 0$ its value is $+l^2 h^3$. For $x = l$ its value is $l^2\,(h_L^{\,3} - h_k^{\,3})$ which will also be positive. This requires that at section L $h_l > h_k$ or the depth at L will be greater than the critical depth.

If the second derivative of h with respect to x be taken, the following expression results:

$$\frac{d^2h}{dx^2} = \frac{h_k^3 h}{(l^2h^3 - h_k^3x^2)^3}\left[\left(6\,h^5x + 6\,\frac{h_k^3}{l^2}\,h^2x^3\right)i - 3\,h^4x^2i^2 - 6\,\frac{h_k^3}{l^2}\,h^3x^2 - 2\,\frac{h_k^6}{l^4}\,x^4 - 2h^6\right].$$

Since i is small, the first term in the bracket will be less, in absolute value, than the sum of the other terms. These being negative, and the fraction before the bracket being positive, since $l^2h^3 - h_k^3x^2 > 0$, the second derivative is always negative: the surface curve is concave with respect to the x axis.

Moreover, since the variation in h is relatively small in practical cases, the absolute value of $\frac{d^2h}{dx^2}$ increases with increasing values of x; that is, the curvature of the surface increases in a downstream direction.

It will now be shown that the slope of the tangent at L' is always greater than the slope of the channel bottom. We have

$$\left(\frac{dh}{dx}\right)_{x=l} = \frac{i\,l^2h_L^3 - 2h_k^3\,l\,h_L}{l^2h_L^3 - h_k^3l^2} < 0$$

which inequality, neglecting the term in i, can only be satisfied when

$$-2\,h_k^3\,l\,h_L < 0.$$

There always exists then, between O' and L' a point P at which the tangent is parallel to the channel bottom.

Hence, the surface curve must have, if the expression $l^2h^3 - h_k^3x^2$ does not vanish for any value of x between zero and l, the shape indicated in figure 20.

It is to be noted that since $h_l > h_k$ the flow downstream of L will be streaming and the water surface elevation L' will be fixed by the conditions of this flow. This is the first case of section 1 of this chapter.

Second case.--Assume now that the denominator of equation (26) will have a root in the interval between zero and l. For $x = 0$, the denominator is positive. Since it vanishes for some value of x between 0 and l, it will be negative for larger values of x. Hence, for $x = l$, we shall have $l^2(h_L^3 - h_k^3) < 0$; that is $h_k > h_l$: the depth at L will be less than critical. This means that downstream of L, where the discharge is constant, there will be shooting flow, as ordinarily defined in hydraulics. But if there is shooting flow to the right of L, the water surface is fixed by conditions upstream of that point. The water surface will be such that the energy line in the reach $0 < x < l$ will have the lowest position possible (case 2, section 1). Now, the position of minimum height is evidently that for which the critical depth is at L. The surface curve will have the shape indicated in figure 21. At L, $h_L = h_k$ and the tangent to the curve, theoretically, would be vertical, since for $x = l$ and $h_l = h_k$ the derivative $\frac{dh}{dx}$ becomes infinite, the denominator being zero. The only root of the denominator admissible is $x = l$; it is seen that this requires shooting flow downstream of L.

Summarizing, there are two possible cases:

First, streaming flow to the right of L. The surface curve can be calculated up to point L', working upstream. At that point, $h_l > h_k$ and the computation for the reach between 0 and l is made by means of formula (19), working upstream from L'.

Second, shooting flow to right of L, which is not affected by conditions to the left of that point. The depth at L is equal to the critical depth, and the computations are made as before, working upstream.

Section 3. Rectangular Channel with Level Bottom, Discharge Increasing in Proportion to the Distance

Make $i = 0$ in equation (26). Then the following differential equation is obtained for the surface curve in a level rectangular channel with lateral inflow which is constant per unit of length:

$$(27) \qquad \frac{dh}{dx} = \frac{-2h_k^3 xh}{l^2h^3 - h_k^3x^2}.$$

This equation can be integrated. Put $x^2 = s$, so that $2xdx = ds$. The equation then becomes, after simplifying:

$$(28) \qquad \frac{ds}{dh} - \frac{s}{h} + \frac{l^2}{h_k^3} h^2 = 0$$

This is a linear equation. Putting $s = uv$ where u and v are new variables, equation (28) becomes:

$$(29) \qquad u\frac{dv}{dh} + v\frac{du}{dh} - \frac{uv}{h} + \frac{l^2}{h_k^3} h^2 = 0.$$

Let u be chosen so that the coefficient of v will be zero, that is:

$$(30) \qquad \frac{du}{dh} - \frac{u}{h} = 0,$$

Equation (29) then becomes:

$$(31) \qquad u\frac{dv}{dh} + \frac{l^2}{h_k^3} h^2 = 0.$$

Equation (30) can be integrated by separation of the variables:

$$\frac{du}{u} = \frac{dh}{h},$$

$$\text{Log } u = \text{Log } h,$$

$$u = h \qquad \text{(constant of integration being unnecessary).}$$

Substituting the value of u in equation (31), it becomes:

$$\frac{dv}{dh} + \frac{l^2}{h_k^3} h = 0.$$

Integrating:

$$v = -\frac{l^2}{2h_k^3} h^2 + c,$$

where c is the constant of integration. Now:

$$s = uv = -\frac{l^2}{2h_k^3} h^3 + ch,$$

and since $s = x^2$, the general integral of (27) is:

$$(32) \qquad x^2 = -\frac{l^2}{2h_k^3} h^3 + ch.$$

The constant c can be determined from the condition (fig.22):

$h = h_l$ for $x = l$, which gives:

$$c = l^2 \left[\frac{1}{h_l} + \frac{h_l^2}{2h_k^3}\right].$$

Substituting this value in equation (32), the equation of the surface curve is obtained:

$$(33) \qquad x^2 = \frac{l^2}{h_k^2} \left[\left(\frac{h_k^2}{h_l} + \frac{h_l^2}{2h_k}\right)h - \frac{1}{2h_k} h^3\right],$$

or

$$(33') \qquad x = \frac{l}{h_k} \sqrt{\left(\frac{h_k^2}{h_l} + \frac{h_l^2}{2h_k}\right)h - \frac{1}{2h_k} h^3}.$$

This equation permits direct computation of as many points as desired on the surface curve by assuming values of h and finding the corresponding values of x.

For $x = 0$ in equation (33), the depth h_o at point O is:

$$(34) \qquad h_o = h_L \sqrt{2 \left(\frac{h_k}{h_L}\right)^3 + 1}.$$

Note that in the special case of a horizontal bottom, the shape of the curve (fig. 22) is the same as for a canal with sloping bottom (fig. 20). The points P and O' are identical.

Section 4. Experimental Verification of Computed Surface Curves for Increasing Discharge

To check the reliability of the methods derived in the first three sections of this chapter, the following program was carried out:

(I) Several surface curves for a rectangular channel, in which the lateral addition to the discharge was made by means of a free jet perpendicular to the main stream, were calculated and determined experimentally.

(II) A series of experiments was made to determine the surface curves in a model of a rectangular escape canal fed by transverse canals. The experimentally determined curves were compared with those calculated.*

*The tests with free jet and escape channel were made by M. Brandle, assistant at the Hydraulic Research Laboratory of Zurich.

(III) Four surface curves were calculated for a trapezoidal channel with increasing discharge, and compared with curves determined experimentally on a reduced model at the Polytechnic School at Brno.

I. EXPERIMENTS ON SURFACE CURVES IN A RECTANGULAR CHANNEL 30 CM WIDE, WITH LATERAL ADDITION TO THE DISCHARGE, MADE BY MEANS OF A FREE JET AT RIGHT ANGLES TO THE MAIN STREAM

Since it is desired to dwell at some length on the description and analysis of the experiments with the escape channel (see sec. 4, II), the practical usefulness of which is beyond doubt, while experiments with the free jet represent more of a theoretical case, only the principal conclusions drawn from the latter will be given here.

These free-jet experiments showed that formulae (18) and (19) give results agreeing with the facts, within the limits of experimental errors. It should be noted that this was because, by using a free jet, it was possible to satisfy rigorously the conditions of per pendicularity of the increase in discharge, a condition assumed in the derivation of the equations.

II. EXPERIMENTS MADE ON A RECTANGULAR CHANNEL 20 CM WIDE, ESPECIALLY BUILT IN THE HYDRAULIC RESEARCH LABORATORY OF ZURICH

(a) Description of the Test Installation.

Figure 24 is a photograph showing a general view of the installation. Figure 23 shows a plan and longitudinal and transverse sections. The installation consists, successively, of:

(1) A gauge box with a calibrated weir fed by a pipe with a regulating valve.

(2) A stilling pool.

(3) A canal with glass sides, 102.7 cm wide, 60 cm deep, and 10.16 m long. The cross section of this canal is divided into two parts. The first part is a model of an open canal 20 cm wide and 30 cm deep, with bottom horizontal. The second part, 73.7 cm by 60 cm, serves only to contain a series of eight small transverse canals 12 cm by 30 cm in cross section and 62 cm long, numbered from I to VIII. These small canals have axes horizontal and perpendicular to that of the main canal, 20 cm wide, into which they empty. The water from the feed canal, 73.7 cm by 60 cm, passes first through risers 12 cm by 20.7 cm, separated from the transverse canals by aerated weirs, without end contractions, and with crests all at the same elevation.

The 20 cm canal represents, then, the upper part of the tailrace canal of a hydroelectric plant, into which eight draft tubes discharge at right angles to the direction of flow.

A needle dam, placed at the downstream end of the main canal, permits regulation of the depth at which the water flows. A second needle dam, placed at the upper end of the channel, permits introduction of any desired discharge at the upstream end.

A coordinatograph, provided with a point gauge, serves to measure water surfaces during the tests.

(4) A receiving tank into which the canal discharges, emptied by a discharge pipe.

(b) Methods and Results of Tests.

1. The first part of the experiments consisted in determining the coefficient of rugosity k of the sides of the model tailrace channel. For this the eight transverse canals were walled off and the upper needle dam left wide open so that the entire discharge over the measuring weir passed through the main channel, the depth of flow being regulated by means of the lower needles. For each condition of flow studied the surface curve was determined by locating, with the coordinatograph, points on the water surface at five cross sections, equidistant along the canal. From the curve thus determined, the slope of the energy line was found which permitted obtaining the value of the coefficient k in Strickler's (Manning's) formula, the discharge being known from the weir measurement.

In this way eight values of k for discharges of 5, 10, 15, and 20 dm^3 per sec. were obtained, two for each discharge. The values of k varied from 100 to 110, with an average value of 105.

2. The weir of transverse canal no. VIII was then calibrated. Water was allowed to flow through this canal, the others remaining closed, the needle dam at the upper end of the main channel also being closed. In this way the weir was calibrated for five discharges of 1, 2, 3, 4, and 5 dm^3 per sec. For each discharge, which was measured very precisely by the calibrated weir in the gauge box, the water-surface elevation in the riser ahead of the weir was determined by means of the coordinatograph. The calibration curve determined by the five points thus obtained is valid for the other seven weirs, these being identical with the one calibrated.

3. A series of experiments was then made with a single transverse canal. Leaving the installation in the condition described, the main feed valve was first opened until the discharge was 10.5 cu. dm per sec. The upstream needle dam was then opened until the water surface elevation in the riser ahead of weir no. VIII corresponded to a discharge of 0.5 cu. dm per sec., which was also, of course, the discharge through the transverse canal. The discharge in the main canal was therefore 10.5 - 0.5 = cu. dm per sec. Thus there existed a steady flow in the main channel of 10 cu. dm per sec. below the canal. By means of the needle dam at the downstream end the depth of water was regulated in the canal so that steady flow at five different stages was obtained, the discharge, however, remaining the same at all stages, that is:

$$\frac{Q_2 - Q_1}{Q_1} = \frac{\Delta Q}{Q_1} = 0.05.$$

For each stage the difference of elevation of the water surface, Δz, above and below transverse canal VIII was measured. This was done by measuring very precisely the water surface elevation at two cross sections; one 33.5 cm upstream of the axis of the transverse canal and one 86.5 cm downstream of the axis. The difference of the averages at each section gave Δz.

Next the same operations were repeated, leaving Q at 10 cu. dm per sec., but making Q_2 successively equal to 10, 11, 12, 13, 14, and 15 cu. dm per sec., corresponding to values of $\frac{\Delta Q}{Q_1}$ equal to 0, 0.1, 0.2, 0.3, 0.4, and 0.5.

The result of all these measurements is given by figure 26, in which are plotted as abscissa the values of h measured at the section upstream of canal VIII defined above, and as ordinates the corresponding values of Δz. The points at the centers of the small circles and triangles show the results of the measurements.

4. After this, experiments were made for a study of the surface curve in a rectangular channel with increasing discharge. For this all the transverse canals were opened, the needle dam at the upper end of the main channel tightly sealed, and that at the lower end removed. Under this condition the entire discharge passed over the eight weirs in the transverse channels, one-eighth of the total passing over each weir (in consequence of the low velocity in the 73.7-cm by 60-cm channel, the water surface in all the risers above the weirs stood at the same elevation).

There was thus obtained in the main canal a flow corresponding to the case in which eight turbines discharge equal quantities of water per unit of time. The surface curves corresponding to discharges of 5, 10, 15, and 20 cu. dm per sec. were then obtained by measuring with the coordinatograph point gauge not only the water surfaces at several sections of the main canal, taking three points at each section, but also the water surface at three points on the axis of each transverse canal. Figure 27a shows, in plan, the location of points for determining the water surfaces. In figure 27b are plotted, as small circles, the average of the points at each section in the main canal and, as horizontal dashes, the average of the points in the transverse canals. Figure 25 shows a view of the reach between canals V and VIII for a discharge of 10 cu. dm per sec.

(c) <u>Discussion of the results of the experiments. Their comparison with those given by computation.</u>

1. The value k = 105 obtained for the coefficient in Strickler's formula is very large. This is not surprising, since one

side of the canal was of glass and the rest of the wetted section had a smooth coating of cement mortar (see transverse section, fig. 23).

2. No remarks are necessary on the calibration of the weirs in the transverse canals.

3. By the aid of formula (19), using $k = 105$, several points on the curve $\Delta z = f(h)$ were calculated for $Q_1 = 10$ cu. dm per sec., and for $\frac{\Delta Q}{Q_1}$ equal to 0, 0.05, 0.10, 0.20, 0.30, 0.40, and 0.50. These curves are plotted with dotted lines in figure 26, which also shows, by full lines, corresponding curves neglecting the effect of friction ($k = \infty$) in formula (19). The comparison of these curves with the plotted points determined experimentally suggests the following remarks:

(a) for $\frac{\Delta Q}{Q_1} = 0$, the experimental points lie within the limits of experimental error, on the dotted curve.

(b) for $\frac{\Delta Q}{Q_1} = 0.05$ and 0.10, the experimental points lie between the curves for $k = 105$ and $k = \infty$.

(c) for $\frac{\Delta Q}{Q_1} = 0.20$, 0.30, and 0.40, the experimental points lie, within the limits of experimental error, on the curve for $k = \infty$.

(d) for $\frac{\Delta Q}{Q_1} = 0.50$, the experimental points lie consistently below the full-line curve.

In a general way, then, it may be seen that the larger the ratio $\frac{\Delta Q}{Q_1}$ the further the experimental points are removed from the dotted curves, $k = 105$. Now, these latter curves ought to correspond exactly to the experimental results. How may the divergence be explained? The explanation is simple. Transverse canal no. VIII which supplies the increment in discharge ΔQ is exactly perpendicular to the main canal. But, by observing carefully the flow lines in the first canal, when they reach the second it will be seen that they curve slightly, several degrees, in the direction of flow of the main stream as if they were influenced by the latter before arriving at the theoretical point of discharge. On this account, the discharge increment does not take place with a component U'' equal to zero, as assumed in equation (19) and as was actually the case in the experiments with the free jet (see above, I) but, on the contrary, with a small positive value of this component. By inspection of equation (14), from which equations (18) and (19) are derived, it is seen that to take account of this condition it is necessary to multiply the term

$\frac{Q_2^2 - Q_1^2}{2g\ S_m^2}$ by the coefficient $(1 - \frac{u^*}{U_m})$. The following equation is then obtained:

$$(19a) \qquad -\Delta z = + \frac{U_m^2}{k^2 R_m^{4/3}} \Delta x + (1 - \frac{u^*}{U_m}) \frac{Q_2^2 - Q_1^2}{2g\ S_m^2} + \frac{U_2^2 - U_1^2}{2g},$$

which would give smaller absolute values of Δz, since u^* and U_m being positive, the coefficient $(1 - \frac{u^*}{U_m})$ is less than one. These new values of Δz should correspond exactly to the facts.

This, then, is the explanation of the divergence between the experiments and equation (19). Some practical observations may now be made. It has been observed that for $\frac{\Delta Q}{Q_1} = 0.2$, 0.3, and 0.4, the experimental points lie, within the limits of experimental error, on the curves computed from equation (19) using $k = \infty$. This is to say that under the experimental conditions when the ratio $\frac{\Delta Q}{Q_1}$ lies between 0.2 and 0.4, the error due to the assumption that the direction of flow of the increment of discharge is perpendicular to that of the main stream is exactly compensated for by the influence of friction.

In the general case, it is evident that the two effects mentioned above, nonperpendicularity of the inflowing stream and roughness of the sides, would not exactly compensate, as happened in some of the experiments. But it may be remarked that these two effects are small and that they always oppose each other. It may be said, then, that by using equation (19), without the first term ($k = \infty$) for the computation of the change of elevation Δz, caused in a tailrace channel by the discharge of a canal perpendicular thereto, a result will be obtained which will, in practice, be satisfactory, and in which the error will certainly be within five percent.

Remarks. The question might be raised as to whether the results obtained might be caused by the distribution of velocities in the transverse canal at the point where it discharges into the main canal. In order to ascertain whether this might be true, obstructions were placed in the transverse canal in such a way as to vary markedly this distribution, and, other conditions remaining the same, values of Δz were measured. The results showed that Δz

was absolutely independent of the velocity distribution in the transverse canal.

4. To compute the surface curves for the experiments in which all eight canals were discharging, use was made of formula (33'), developed in section 3 of this chapter. This formula neglects the influence of friction, and it has just been shown above (3) that this influence, in the problem in hand, is more or less compensated for by the nonperpendicularity of the flow lines of the discharge increments to those of the main channel. Formula (33') assumes, also, that the lateral increment is uniformly distributed along the reach over which the discharge varies. This hypothesis is admissible, being analogous to that made in strength of materials when it is assumed that a beam subjected to flexure is uniformly loaded, while in the majority of practical cases the load is, in reality, concentrated at points close together.

The four surface curves, computed by formula (33'), are shown in figure 27b. It is seen that they correspond very well with the results of measurement shown by the plotted points except, perhaps, in the vicinity of transverse canals VII and VIII where local surface disturbances exist, effect of which can evidently not be included in computations.

The water surfaces in the transverse canals (shown by horizontal dashes in figure 27b) are, in general, at the same elevations as the corresponding points on the surface curve. There is, nevertheless, an exception to this rule in the case of canals VII and VIII where they are a little higher. This is also due to local disturbances, produced by the strong current in that zone.

Summarizing, it is seen that under the conditions of the experiment, formula (33') gives exact results, to within a few percent, the influence of local disturbance due to the relatively high velocities evidently always being neglected. In a concrete case, if the above remarks (3) on the subject of friction and nonperpendicularity be recalled, it may be said that formula (33') will give, when applied to the tailrace of a power plant, results accurate to at least five percent.

III. COMPARISON OF FOUR COMPUTED SURFACE CURVES FOR A TRAPEZOIDAL CHANNEL WITH INCREASING DISCHARGE WITH EXPERIMENTS MADE ON A REDUCED MODEL AT THE BRNO LABORATORY

Professor Smrcek, in the laboratory of the polytechnic school at Brno, made a study of the spillway channel of Tieton dam, in the state of Washington, on a model on 1:50 scale. The result of this study was published in "Comptes rendus du XVme Congrès international de Navigation, en 1931."*

*Rapport de MM. Smrcek et Smetana, No. 63 lere section, 3me communication.

On plate II of that publication are shown longitudinal and transverse sections of the channel as well as surface curves for model discharges of 48.05, 64.07, 80.02, 111.10 cu. dm per sec. Figures 10 and 12 of that plate are reproduced here, figure 28. It is seen that in this design the channel receives the discharge from six spillway gates. The water enters almost at right angles to the direction of flow in the channel.

The four surface curves have been computed, beginning with the results of the experiments. This was done as follows:

(1) That part of the channel over which the discharge varies was first divided into six equal parts, Δx, corresponding to the six gates.

(2) Next, the lateral increment in discharge for each of the six sectors was determined. For flows of 48.05 and 64.07 cu. dm per sec. it was assumed that the gate discharge was not influenced by the water in the channel, which is equivalent to assuming that the discharge of all six sectors is equal. For the flow of 80.02 cu. dm per sec. it was assumed that the surface curve affected the discharge of the first three gates, and for the flow of 111.10 cu. dm per sec. it was assumed that five of the six gate openings were partially submerged. In the two latter cases, then, there existed unequal discharges through the different sectors. The discharges were computed by means of the formula for submerged weirs.

(3) Next, the critical depths at section 3', at the downstream end of the section over which the discharge varies, were computed for each discharge.

(4) The water-surface elevations thus obtained served as starting points for the computation of the surface curves, using formula (19) with $k = \infty$. It would have been, in fact, difficult to introduce the effect of friction, the roughness of the sides being unknown. Moreover, this effect should, in this case, be partially compensated for since the channel has a very steep slope, about 10 percent, which gives to the inflow a slight velocity component, U^*, in the direction of flow in the channel. The effect of this component, as in the case of the tailrace, is to decrease the absolute value of the second term of equation (19). For exact results, assuming the value of k to be known, the effect of U^* could be included, by use of formula (19a),

but the results would certainly differ very little from those obtained.

Equation (19) was then applied, successively, four times, using $k = OO$, working upstream. There were thus obtained seven points for each surface curve. These curves are shown on figure 28.

It is seen that the agreement between the computed and experimental curves is good.

Section 5. Conclusions - First Part of Chapter

The several experiments given confirm the theoretical considerations developed in the first three sections of this chapter. When the lateral increment is exactly perpendicular to the principal current, the equations established are exact (case of inflow by jet). In the case of a spillway or a tailrace it has been shown that perpendicularity does not exist, but that the error involved in the assumption is practically compensated for if friction be neglected. In this way results are obtained in which the error does not exceed five percent.

Finally, it should be noted that if the inflow falls a considerable distance, the main stream could contain a large percentage of air, of which the computations evidently take no account. In this case the formulae and conclusions developed in this chapter will not apply with the same exactness.

SECOND PART

SURFACE CURVES FOR DECREASING DISCHARGE, WITH LATERAL SPILL WHOSE VELOCITY COMPONENT IN THE DIRECTION OF FLOW OF THE MAIN STREAM IS ASSUMED EQUAL TO THE MEAN VELOCITY OF THAT STREAM

Section 6. Calculation of Surface Curve for Decreasing Discharge in a Channel of Any Section

The following assumptions will be made:

(1) The discharge decreases in a downstream direction ($dQ < 0$). This decrease takes place according to a known law (for example, outflow over a side weir).

(2) At the point of the cross section at which the water leaves the channel to flow over the side weir, the velocity component U^*

parallel to the direction of the main stream is equal to the mean velocity U of that stream.

(3) The variation $d\alpha$ of the momentum coefficient will be neglected, and, for simplicity, α will be assumed equal to unity, $\alpha = 1$.

Under these conditions, equation (14) becomes:

$$(35) \qquad dz = -\frac{U^2}{g}\left[\frac{g}{k^2R^{4/3}}\,dx + \frac{dU}{U}\right],$$

and if this be placed in the form for finite differences, we obtain, using the notations of paragraph 1 for equation (19) (see fig. 29):

$$(36) \qquad -\Delta z = +\frac{U_m^{\,2}}{k^2R_m^{\,4/3}}\,\Delta x + \frac{U_2^{\,2} - U_1^{\,2}}{2g}. \text{*}$$

---

* Professor Engels has derived, purely by experiment, the formula $-\Delta z = 1.087\,\frac{U_2^{\,2} - U_1^{\,2}}{2g}$ (Mitteilungen aus dem Dresdener Flussbau-Laboratorium, Heft 200 et 201, Berlin V.D.I. 1917). This formula corresponds to equation (36) if friction is neglected and the second term modified by a coefficient $\alpha = 1.087$. It may therefore be said that the experiments of Engels confirm the equation.

This equation has the same form as equation (16) used in the calculation of surface curves for constant discharge. There exists nevertheless, an essential difference between these two equations. While in equation (16) the mean velocities U_1 and U_2 satisfy the equation of continuity

$$U_2S_2 - U_1S_1 = 0,$$

in equation (36) they must satisfy the relation

$$(37) \qquad U_2S_2 - U_1S_1 = \Delta Q.$$

If H_1 and H_2 be the heads on the discharge weir at sections 1 and 2, H_m the mean head between these sections, $\mathcal{V}$ coefficient dependent on the type of weir, the discharge, ΔQ may be expressed as:

$$\Delta Q = -\psi H_m^{3/2} \Delta x. \tag{38}$$

The three equations (36), (37), and (38) permit calculation, by finite differences, of the surface curves, starting from a known point.

The method of procedure is as follows: Assume as known all the quantities affecting flow at point P_1 (S_1, R_1, Q_1, U_1, H_1) and that it is desired to find these quantities at point P_2. Assume first an arbitrary value of Δz, from which S_2, R_2, H_2 and consequently S_m, R_m, and H_m can be determined. Equation (38) will then give ΔQ, and substituting its value in (37), U_2 will be obtained. All the terms in the second member of equation (36) are then known, which permits calculation of Δz. If the value found agrees with that assumed, the calculation is finished. If not, it will be necessary to continue until the two values agree.

Use of formulae (36), (37), and (38) offers no difficulty. As for the surface curves for decreasing discharge, two cases may be distinguished, and there should be repeated here, word for word, what was said at the end of paragraph (section) 1 of this chapter.

CHAPTER IV

CONCLUSIONS

I. The variation of pressure $dz = \frac{dp}{\Pi}$ and the loss of energy dH_e between two adjacent sections of a stable stream (courant liquide stationnaire) are given by the two general equations:

$$(14)\quad dz = \frac{dp}{\Pi} = \frac{U^2}{g}\left[-\frac{g}{k^2R^{4/3}}\,dx - d\alpha - \left(\alpha - \frac{u^*}{U}\right)\frac{dQ}{Q} - \alpha\frac{dU}{U}\right],$$

$$(15)\quad dH_e = \frac{U^2}{g}\left[-\frac{g}{k^2R^{4/3}}\,dx + \frac{1}{2}\,d\alpha - \left(\alpha - \frac{u^*}{U}\right)\frac{dQ}{Q}\right].\ [1]$$

---

[1] If, in those formulae, the meter, the ton, and the second be adopted as units, dz represents the variation in pressure, expressed in meters of water (m); dp the same variation, but expressed as t/m^2; Π the unit weight of the water (t/m^3); dx the distance between two sections (m); U the mean velocity at a section (m/sec); dU the change in this velocity from one section to the other (m/sec); Q the discharge at a section (m^3/sec); dQ the change in discharge from one section to the other (m^3/sec); u^* the component, parallel to the main stream, of the velocity of arrival or departure of the exterior increment dQ(m/sec); α the momentum coefficient at a cross section (pure number); $d\alpha$ the change in this coefficient from one section to another; R the hydraulic radius at one section (m); g the acceleration of gravity (m/sec^2); k the coefficient, dependent on the roughness of the sides of the channel, in Strickler's formula ($m^{1/3}$/sec).

II. These equations show that the variation in pressure dz is proportional to the square of the mean velocity U and depends on the four following causes:

(1) Roughness of the sides (k),

(2) Variation $d\alpha$, from one section to the next of the form of the velocity diagram,

(3) Variation in discharge dQ,

(4) Variation in mean velocity dU.

VIII. Professor Smreek, at the Polytechnic School at Brno, made experiments on a steeply sloped trapezoidal channel with lateral inflow at right angles to the current. Beginning with these experiments, surface curves have been computed by means of formula (19). The agreement of calculation and experiment is good.

IX. Finally, equation (14) allows derivation of a formula for finite differences for use in calculating surface curves in open channels with lateral outflow, of which the velocity component, parallel to the direction of the channel, is equal to the mean velocity in the channel.

$$(36) \qquad -\Delta z = + \frac{U_m^2}{k^2 R_m^{4/3}} \Delta x + \frac{U_2^2 - U_1^2}{2g} .$$

In this formula notations are the same as for equation (19). Experiments by Professor Engels in 1917 confirm the validity of equation (36).

Zurich, September 14, 1932.

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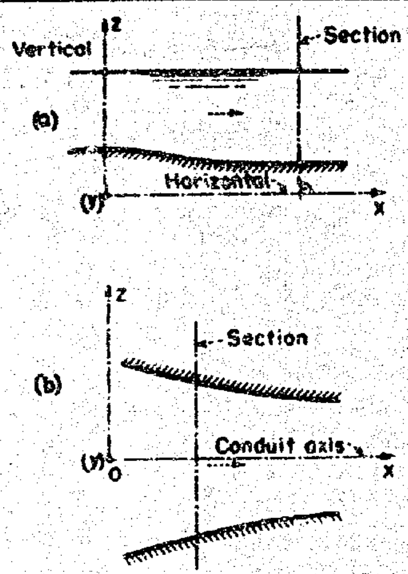


FIG. 1 (a) OPEN CHANNEL
(b) PRESSURE CONDUIT

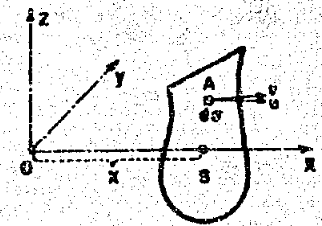


FIG. 2 CROSS SECTION OF FLOWING STREAM

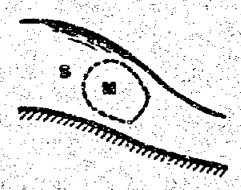


FIG. 3 LIQUID MASS M WITH SURFACE BOUNDARY S

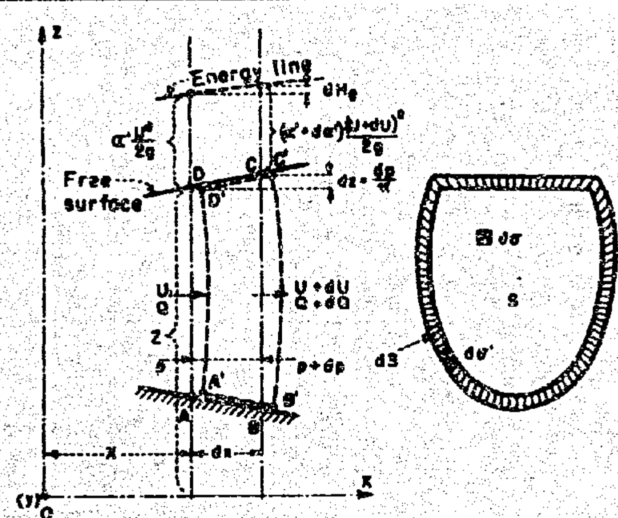


FIG. 4 OPEN CHANNEL

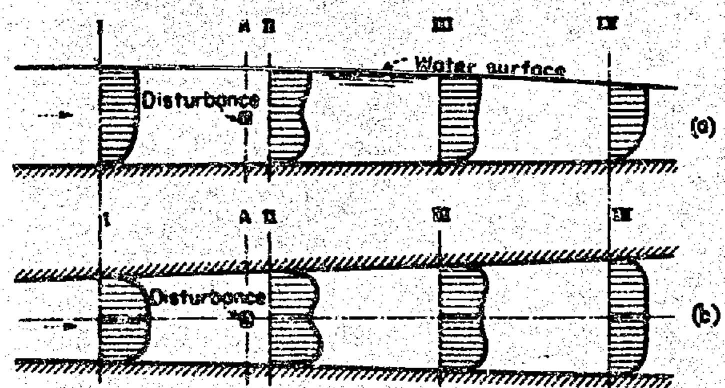


FIG. 6 ALTERATION OF VELOCITY DISTRIBUTIONS FOLLOWING
A DISTURBANCE (a) OPEN CHANNEL (b) PRESSURE CONDUIT

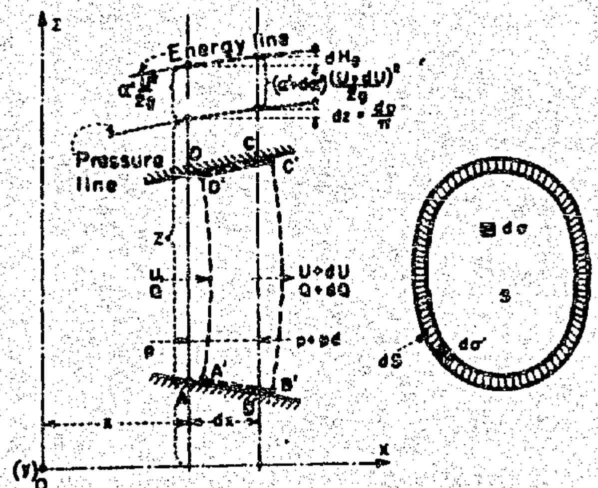


FIG. 5 PRESSURE CONDUIT

X-D-1996

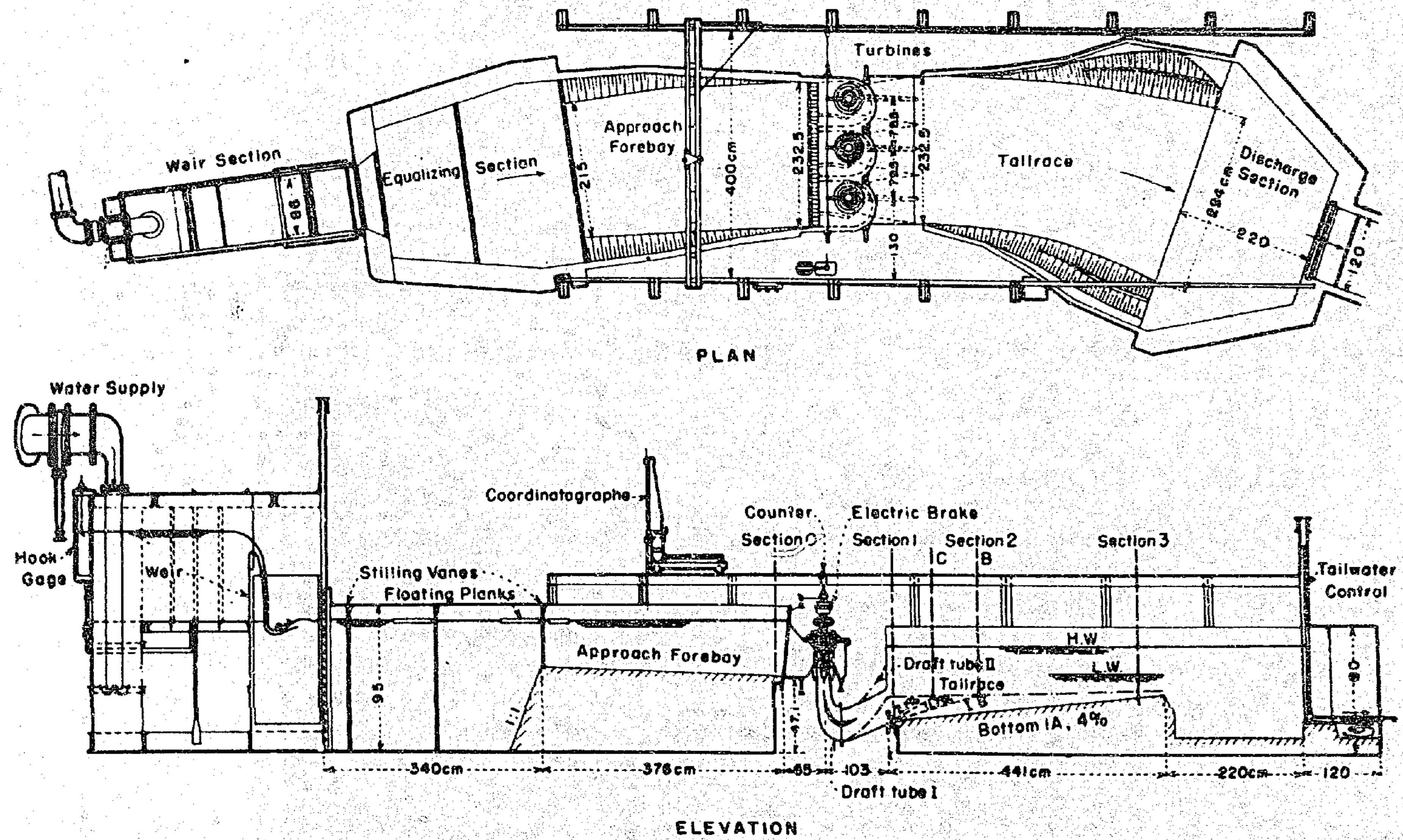


FIG. 9 PLAN AND SECTION OF MODEL FOR TESTING TAILRACE OF DOGERN POWER PLANT

X-D-1987

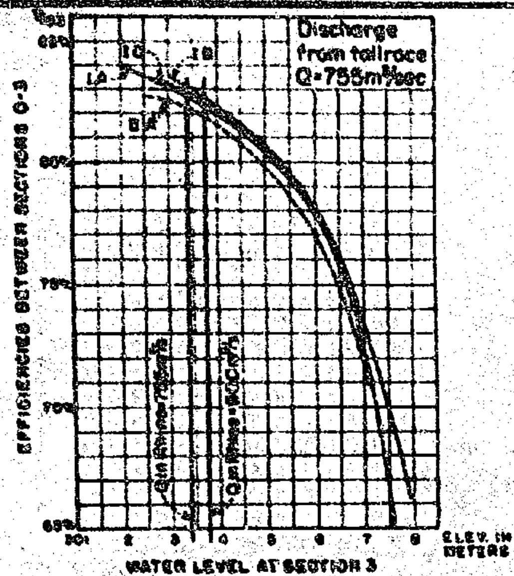


FIG. 10 EFFICIENCY DIAGRAM FOR DOGERN PLANT BETWEEN SECTIONS O-3 FOR LAYOUTS IA, IB, IC, AND IIA

(Note: Streamflow plant; Section O in Forebay)

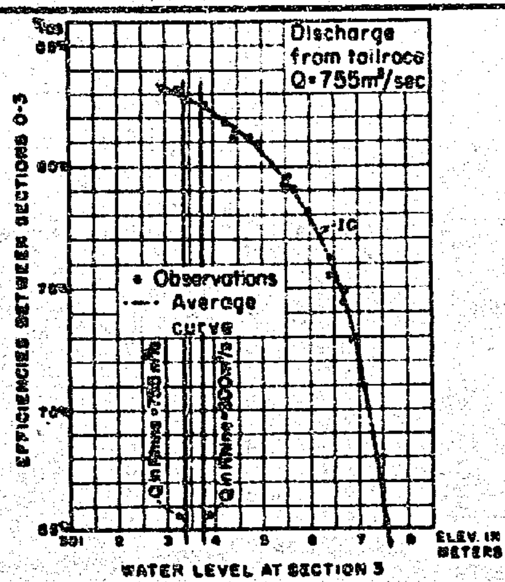


FIG. 11 EFFICIENCY DIAGRAM FOR DOGERN PLANT BETWEEN SECTIONS O-3 FOR LAYOUT I C

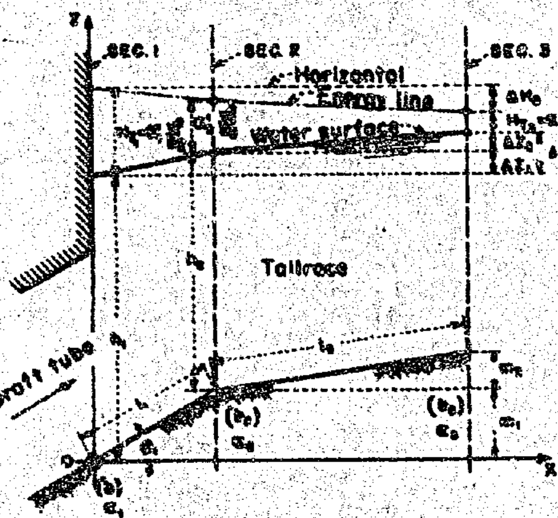


FIG. 13 LONGITUDINAL SECTION OF TAILRACE

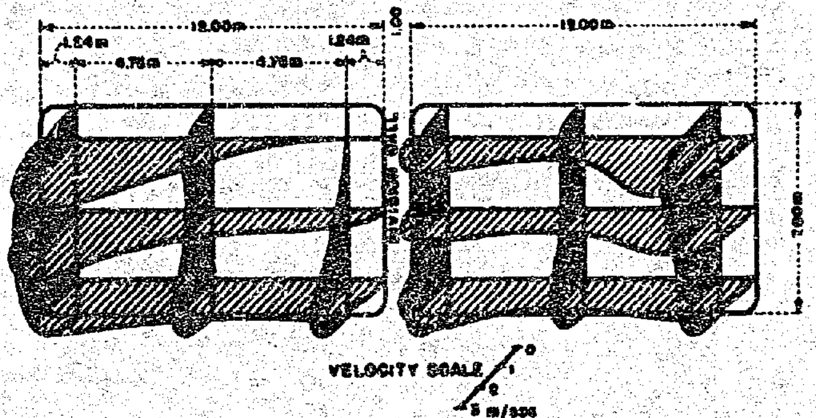


FIG. 14 TRIAXIAL DIAGRAM OF VELOCITY DISTRIBUTION AT EXIT FROM DRAFT TUBE FROM RIGHT HAND UNIT. DRAFT TUBE CONTAINS VERTICAL DIVISION WALL EXPERIMENT IA; Q=755m³/sec

X-D-1939

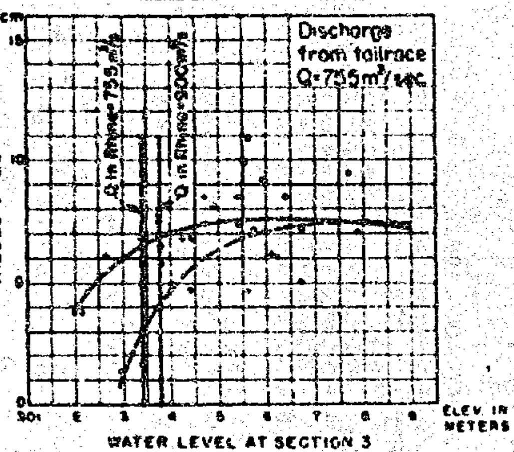


FIG. 15 VARIATIONS IN THE WATER LEVEL ΔZ BETWEEN SECTIONS 1-3 AS A FUNCTION OF LEVELS AT SECTION 3

CALCULATED CURVE - LAYOUT IA
CALCULATED CURVE - LAYOUT IC
TESTS LAYOUT IA
TESTS LAYOUT IC

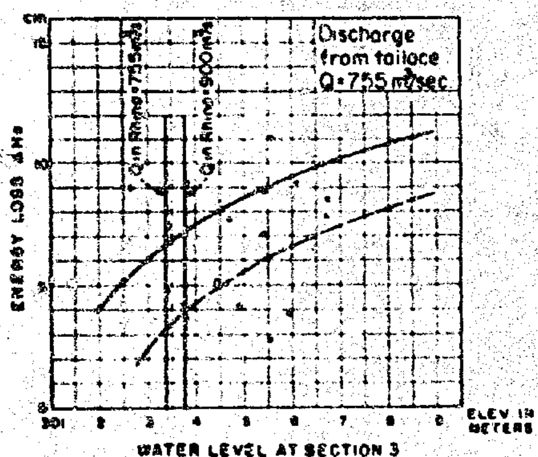


FIG. 16 ENERGY LOSS ΔH_e IN THE TAILRACE AS A FUNCTION OF THE LEVEL AT SECTION 3

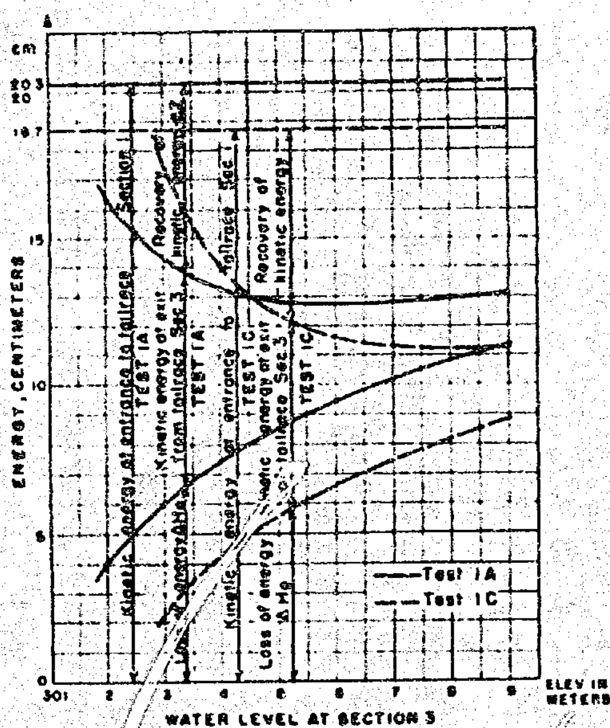


FIG. 17 DIAGRAM SHOWING VARIATIONS IN ENERGY IN TAILRACE FOR TESTS IA AND IC FOR A DISCHARGE OF 755 m³/sec

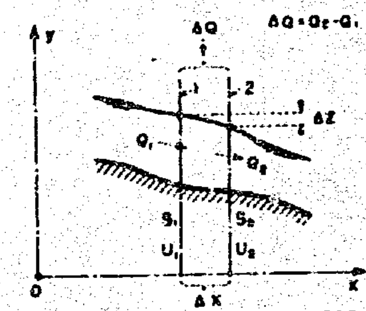


FIG. 18 OPEN CONDUIT WITH LATERAL INFLOW

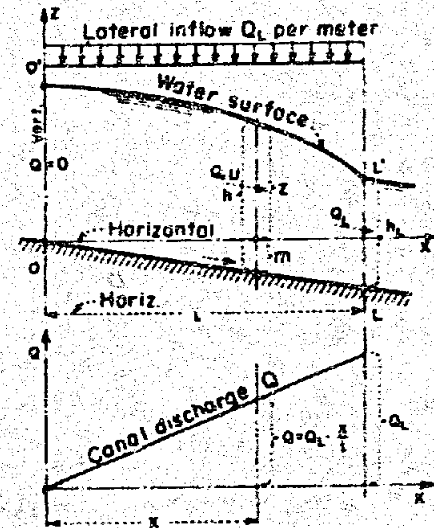


FIG. 19 LONGITUDINAL SECTION OF RECTANGULAR CANAL WITH CONSTANT FALL AND WITH CONSTANT INFLOW PER METER

X-D-1990

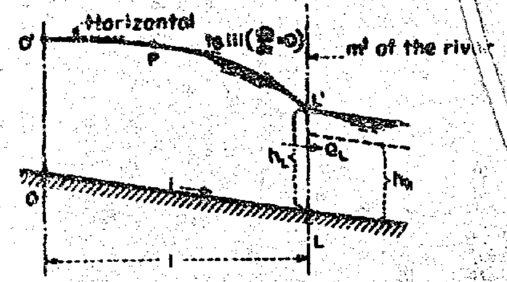


FIG. 20 SURFACE CURVE FOR A RECTANGULAR CANAL WITH CONSTANT LATERAL INFLOW PERPENDICULAR TO THE STREAM 1st CASE; $h_L > h_E$ THE FLOW IS CONTROLLED BY CONDITIONS DOWNSTREAM FROM SECTION L

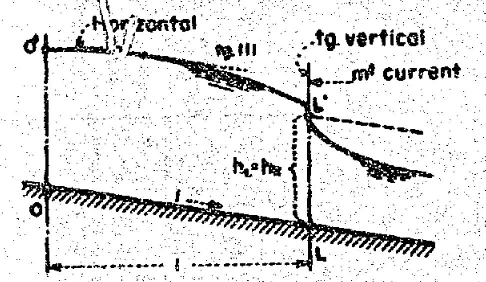


FIG 21. SURFACE CURVE FOR A RECTANGULAR CANAL WITH CONSTANT LATERAL INFLOW PERPENDICULAR TO THE STREAM 2nd CASE; $h_L = h_E$, THE FLOW IS CONTROLLED BY SECTION L

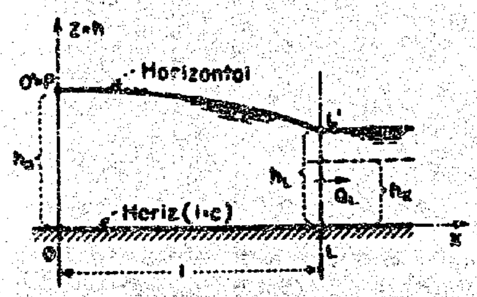


FIG 22 SURFACE CURVE IN A RECTANGULAR CANAL WITH LEVEL BOTTOM WITH CONSTANT INFLOW PERPENDICULAR TO THE STREAM

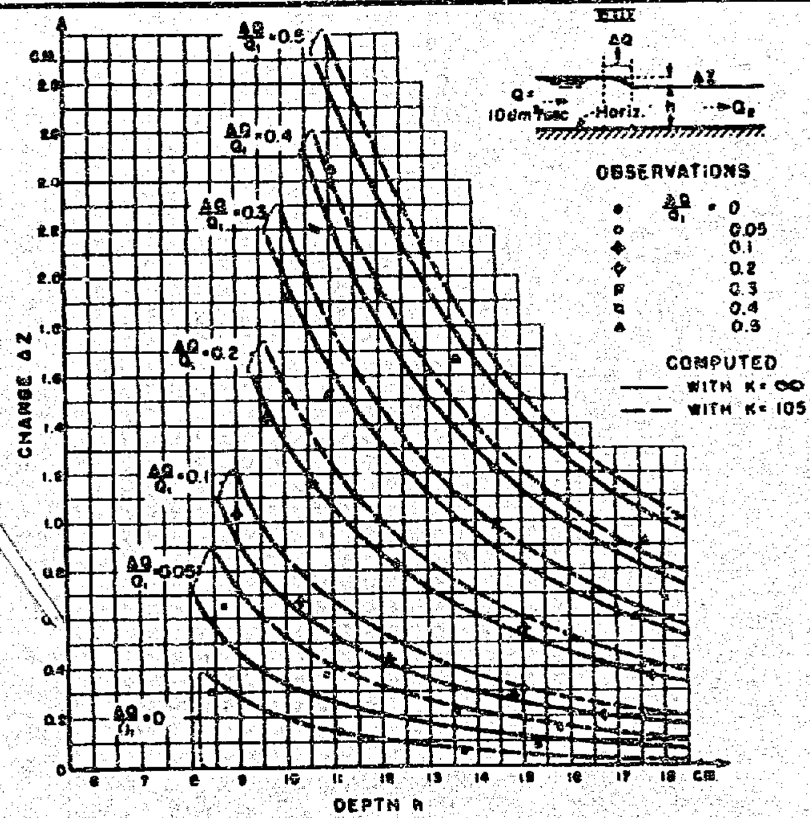


FIG. 28 OPEN RECTANGULAR CANAL VARIATIONS IN LEVEL OF SURFACE CURVE ΔZ AS A FUNCTION OF DEPTH h FOR DIFFERENT LATERAL INFLOWS ΔQ COMPARISON OF OBSERVED AND COMPUTED VALUES (EQUATION 19)

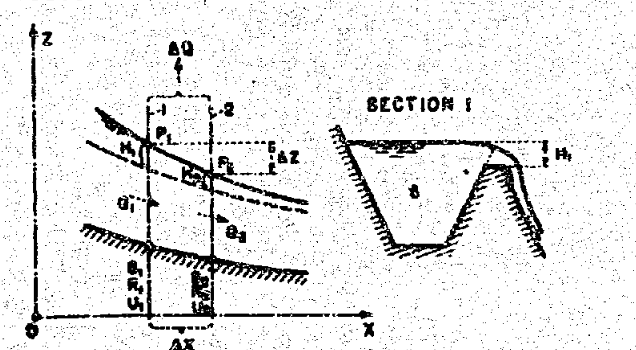
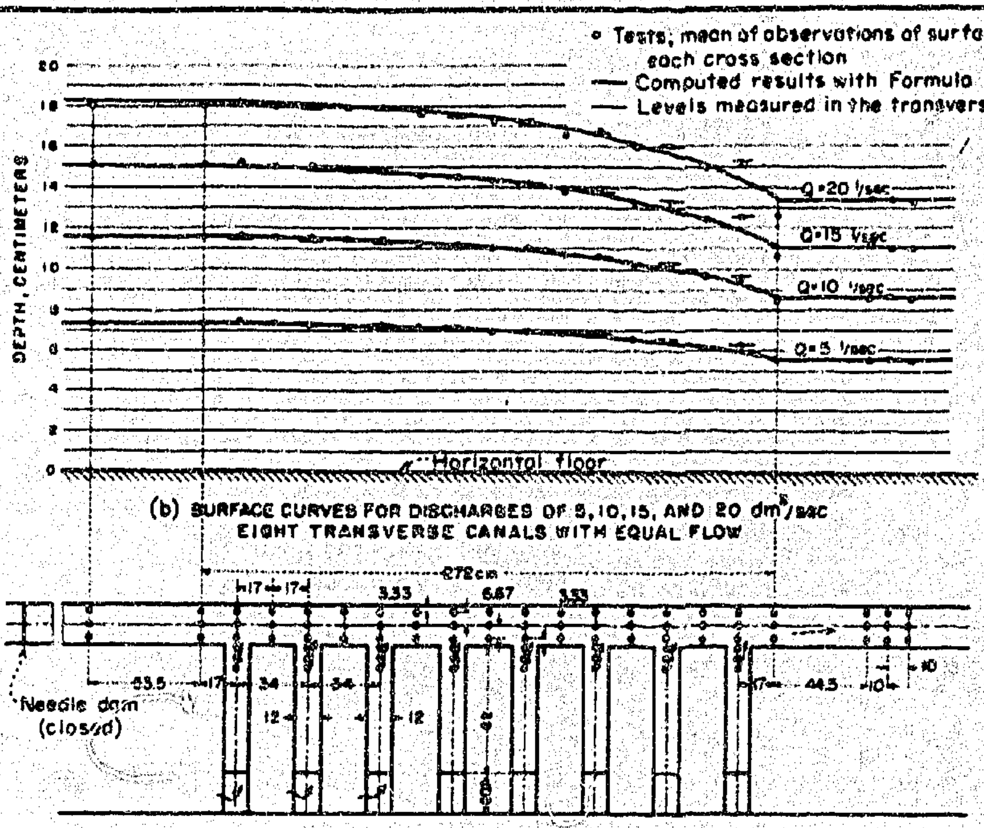


FIG. 29 OPEN CHANNEL OF ANY SECTION WITH SIDE SPILL

X-D-1891



PLAN VIEW SHOWING LOCATIONS OF MEASUREMENTS

FIG 27 RECTANGULAR CANAL BED WITH EIGHT TRANSVERSE CANALS
SURFACE CURVES ON THE MAIN CANAL

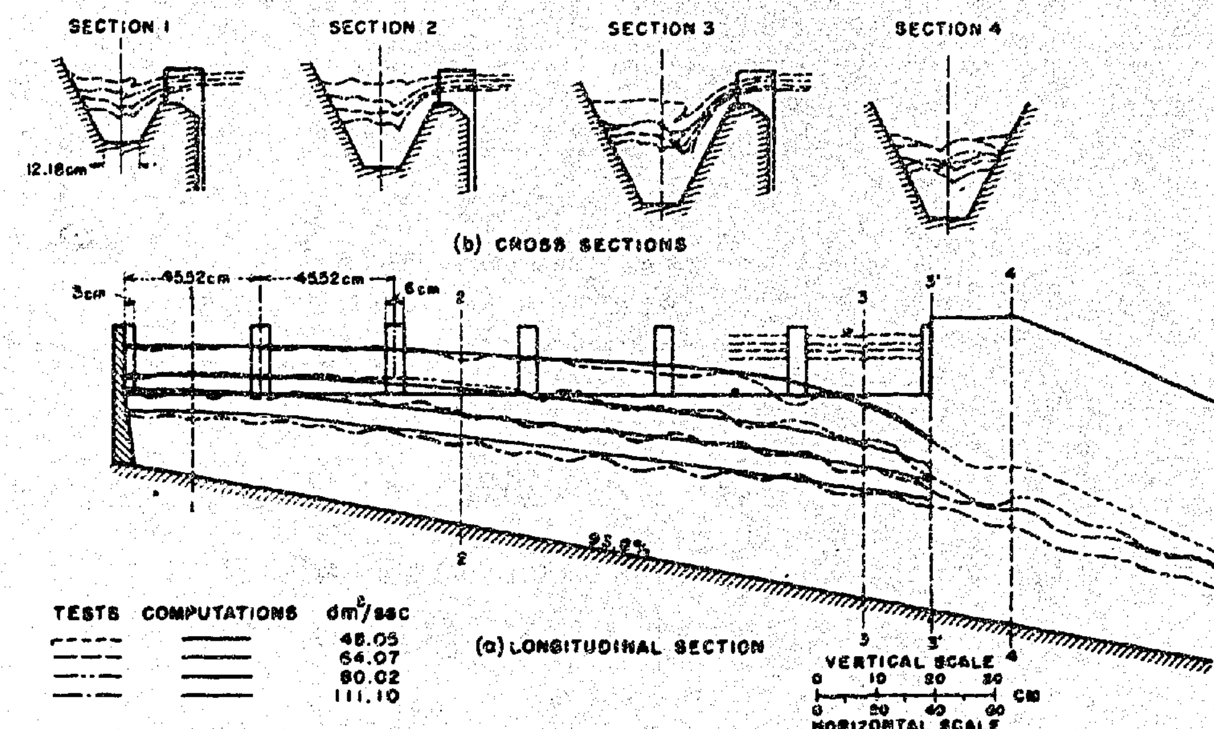


FIG. 28 SPILLWAY FOR THE TIETON DAM, WASHINGTON U.S.A. TESTS ON SMALL MODEL OF SPILLWAY. COMPARISON OF SURFACE CURVES OBTAINED BY THE TECHNICAL UNIVERSITY AT BRNO WITH THE CALCULATIONS MADE BY THE HYDRAULIC RESEARCH LABORATORY OF ZURICH

X-D-1988