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THEORY AND PROBLEMS OF WATER PERCOLATION

By Carl N. Zangar

Denver, Colorado
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L. N. McCLELLAN, Chief Engineer

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THEORY AND PROBLEMS OF WATER PERCOLATION

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Engineering Monograph No. 8

Theory and Problems of Water Percolation

ERRATA

Page 3, figure 1:
The upper coordinate system should bear the subtitle,
a. Cylindrical coordinates (r,θ,z).

Page 11, next to last line of right hand column:
$Q = 0.006{,}996$.

Page 14, figures 11 and 12, and page 15, figure 13:
Ordinates should be c rather than k; statement in upper right of each figure should read "Values of c for values of ..." rather than "Values of k for values of ..."

Page 52, figure 43:
Ordinates should be simply $\frac{h_1}{r_1}$, not "HEAD ON WELL BOTTOM - $\frac{h_1}{r_1}$ - RADII."

Page 54, figure 45:
Values of C_u are from figure 43 (not figure I), and $\frac{L_A}{h_1} = 1.00$, (not $\frac{L_A}{h_1}$ - 1.00).

Page 62, figure 53:
The phrase, (steady state), should follow $Q = 0.50$ ft.^{3}/sec.

ERRATA--ENGINEERING MONOGRAPH No. 8

THEORY AND PROBLEMS OF WATER PERCOLATION

by Carl N. Zangar

p. 68. . . Eq. (14A) should read:

$$H = \frac{Q}{4\pi K L}\left[\ln \frac{\frac{c-b}{2} + \sqrt{r^2 + \left(\frac{c-b}{2}\right)^2}}{\frac{b-c}{2} + \sqrt{r^2 + \left(\frac{b-c}{2}\right)^2}} - \ln \frac{\frac{b+3c}{2} + \sqrt{r^2 + \left(\frac{b+3c}{2}\right)^2}}{\frac{c+3b}{2} + \sqrt{r^2 + \left(\frac{c+3b}{2}\right)^2}}\right]$$

p. 68. . . last paragraph in 2nd column should read:

Experimental results are in almost complete agreement with this analysis. Results of electric analogy tests show average values not greater than 12 percent below calculated values for $L/r \geqq 8$. The approximate mathematical method has reasonable validity for $L/r \geqq 5.0$, and almost perfect agreement for $L/r \geqq 20$.

p. 70. . . Eq. (7B) should read:

$$p = \frac{Q}{2\pi H^2 K}\left[(H-y)\sinh^{-1}\frac{(H-y)}{x} + (H-y)\sinh^{-1}\frac{y}{x} - \sqrt{x^2 + (H-y)^2} + \sqrt{x^2 + y^2}\right]$$

p. 71. . . Eq. (8B) should read:

$$p_o = \frac{Q}{2\pi H^2 K}\left[H \sinh^{-1}\left(\frac{H}{a}\right) - \sqrt{a^2 + H^2} + a\right]$$

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and so

$$\left.\begin{aligned} v_x &= K\frac{\partial \phi}{\partial x} \\ v_y &= K\frac{\partial \phi}{\partial y} \\ v_z &= K\frac{\partial \phi}{\partial z} \end{aligned}\right] \quad \ldots\ldots\ldots\ldots (9)$$

For incompressible liquids the equation of continuity holds, so we may write

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad . (10)$$

and substitution of equations (6) and (9) into (10) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi =$$

$$\nabla^2 p = 0 \quad \ldots\ldots\ldots (11)$$

Equation (11) is Laplace's equation in three dimensions. Any function p or ϕ that satisfies Laplace's equation is a solution to a flow problem if the boundary conditions can be satisfied. Equations similar to (11) govern the steady flow of heat and electricity. It is for this reason that the electric analogy may be used to solve problems in the steady flow of fluids.

The pressure function that satisfies equation (11) is known as the potential function. It, of course, must satisfy the boundary conditions and since it was derived from Darcy's law it is subject to the same restrictions. The potential function, since it applies to the steady state, is based on the assumption that the soil mass contained in the flow system is completely saturated.

It is possible at this point to state certain boundary conditions in terms of the ϕ function. For example:

1. At an impermeable boundary

$$\frac{\partial \phi}{\partial n} = 0$$

where n is normal to the boundary.

2. For a constant potential surface

$$\phi = \text{constant.}$$

3. For a free surface (such as the phreatic line in an earth dam, or a streamline and a constant-pressure line),

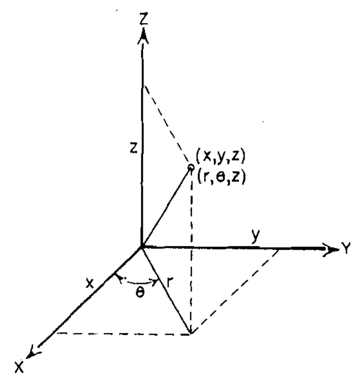
$$\frac{\partial \phi}{\partial n} = 0 \text{ and } p = C = \phi - \frac{g_z z}{\gamma}$$

4. For a seepage surface (a constant-pressure surface, but not a streamline),

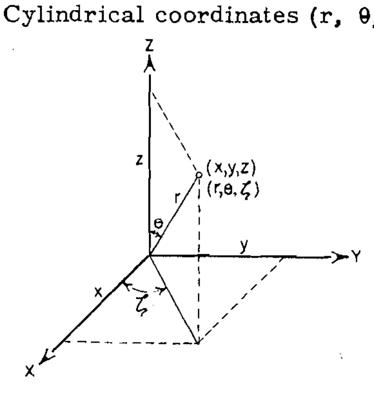
$$\phi - \frac{g_z z}{\gamma} = p = C$$

There are many fluid systems that possess axial symmetry and for these problems it is convenient to express Laplace's equation, equation (11), in cylindrical coordinates (r, θ, z), (see figure 1a). The velocity components become

$$\left.\begin{aligned} v_r &= K\frac{\partial \phi}{\partial r} \\ v_\theta &= \frac{K}{r}\frac{\partial \phi}{\partial \theta} \\ v_z &= K\frac{\partial \phi}{\partial z} \end{aligned}\right] \quad \ldots\ldots\ldots\ldots (12)$$



a. Cylindrical coordinates (r, θ, z).



b. Spherical coordinates (r,θ,ζ).

Figure 1 - Two Coordinate Systems.

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INTRODUCTION

The flow of water through dams and their foundations, and the accompanying pressures and gradients that exist, have long been recognized by engineers as important factors in dam design. This monograph is concerned with the effects of this "percolating" water and the methods for correcting these effects when they are thought to be detrimental. Also given are several methods for determining the permeability of soils by field tests.

These problems resolve themselves into a study of the slow flow of water through porous media. Slow flow as used here is defined as laminar flow in which the Reynolds number is 1 or less. If the Reynolds number becomes larger than 1, it is possible for turbulence to develop. In this case, Darcy's law governing the slow flow of water through porous media, no longer applies. Darcy's law will be treated in detail under the section on general theory, which follows; but briefly, it states that the rate of flow, Q, of water through a porous medium is directly proportional to the cross-sectional area, A, and to the pressure gradient acting.

There are many engineering problems to which the laws of slow flow of water apply and which, consequently, affect the design of the structures involved. Some of these problems are:

1. Percolation through concrete dams and their foundations.

2. Percolation through earth dams and their foundations.

3. Flow into drains embedded in concrete and soil.

4. Flow around cut-off walls.

5. Foundation settlement (consolidation).

Most of these problems involve a knowledge of the permeability of the materials involved.

Percolating water, while not necessarily dangerous, usually results in one or more of the following objectionable conditions:

1. Water losses by seepage through the dam and foundation.

2. Uplift pressures that tend to cause overturning of the dam.

3. Flotation gradients (piping) that may cause local failure or even total failure of a structure.

4. Application of body forces which affect stability.

There are several methods which may be used to assist in solving the problems encountered as a result of percolating water. These methods include pure mathematics, electric or membrane analogy experiments, hydraulic model experiments, and field experiments. Solutions to some flow problems may be obtained by a combination of methods, as, for example, the combination of an electric analogy experiment with a hydraulic model experiment.

GENERAL THEORY

The movement of water through granular materials was first investigated by Darcy in 1856 when he became interested in the flow characteristics of sand filter beds.[1] In his experiments he discovered the law governing the flow of homogeneous fluids through porous media. Darcy's law is expressed by the equation

$$Q = \frac{KAH}{L} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

Q = rate of flow,
A = cross-sectional area,
H = head,
K = permeability coefficient, and
L = length of path of percolation.

Many experimenters have worked on the range of validity of Darcy's law and their results are not in complete agreement. But all have expressed the applicable range in terms of Reynolds number, which is well known in hydraulics and hydrodynamics. The Reynolds number is given by the equation

$$R = \frac{d\, v\, \gamma}{\mu} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which

R = Reynolds number,
d = diameter of the average grain,
v = average velocity of flow, through the pores,
γ = density of water, and
μ = absolute viscosity of water.

[1] Darcy, H., Les Fontaines Publiques de la Ville de Dijon, Dalmont, Paris, 1856.

The diameter of the average grain used in equation (2) is defined by the relation

$$d = \sqrt[3]{\frac{\Sigma n_s d_s^3}{\Sigma n_s}} \quad \ldots\ldots\ldots \quad (3)$$

in which

d_s = arithmetic mean of the openings in any two consecutive sieves of the Tyler or U. S. Standard sieves, and
n_s = number of grains of diameter d_s found by a sieve analysis.

Physically, d should represent the average pore diameter rather than the diameter of the average grain. However, the average pore diameter can be measured directly only by microscopic examination of a cross-section of the porous medium itself. Therefore, in the case of soils, all attempts to define or use a value of d in Reynolds number have referred to the diameter of the average grain.

For the above definition of Reynolds number, experimenters [2,3] have determined that Darcy's law holds only if the relation $R \leqq 1$ is satisfied.

The general differential equation for the flow of water through homogeneous porous media is readily deduced from the generalized form of Darcy's law and the equation of continuity.

From Darcy's law, equation (1), and the principle of dimensional homogeneity, it can be shown that

$$v = C \frac{d^2}{\mu} \frac{dP}{ds} \quad \ldots\ldots\ldots\ldots \quad (4)$$

where

C = a dimensionless constant,
P = pressure,
s = length along path of flow, and
$\frac{dP}{ds}$ = pressure gradient.

d^2, μ, and C may be grouped to make one constant K, the familiar coefficient of permeability, if we express the gradient in terms of pressure head p, instead of pressure P. (K will be constant for any particular material if the temperature does not change. Since the viscosity of water varies appreciably with temperature, any considerable temperature variation may warrant a corresponding modification of K.)

We may then write

$$v = K \frac{dp}{ds} \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

where

$\frac{dp}{ds}$ = hydraulic gradient.

Now consider the case of three-dimensional flow and assume that the resultant fluid velocity given by equation (5) may be resolved into three components along the selected coordinate axes. Then if K has different values along the coordinate axes, Darcy's law may be written as

$$\left.\begin{aligned} v_x &= K_x \frac{\partial p}{\partial x} \\ v_y &= K_y \frac{\partial p}{\partial y} \\ v_z &= K_z \frac{\partial p}{\partial z} \end{aligned}\right] \quad \ldots\ldots\ldots \quad (6)$$

If the fluid is of specific weight, Υ, and there exists a body force of components g_x, g_y, and g_z per unit of volume acting on the fluid, it will affect the velocity just as the hydraulic gradients do and equation (6) becomes

$$\left.\begin{aligned} v_x &= K_x \left(\frac{\partial p}{\partial x} + \frac{g_x}{\Upsilon}\right) \\ v_y &= K_y \left(\frac{\partial p}{\partial y} + \frac{g_y}{\Upsilon}\right) \\ v_z &= K_z \left(\frac{\partial p}{\partial z} + \frac{g_z}{\Upsilon}\right) \end{aligned}\right] \quad \ldots\ldots\ldots \quad (7)$$

When the positive Z axis is taken upward, K assumed independent of direction, and gravity the only body force, the potential function ϕ may be written as

$$\phi = p + \frac{g_z z}{\Upsilon} \quad \ldots\ldots\ldots\ldots \quad (8)$$

[2] Fancher, G. H., Lewis, J. A., and Barnes, K. B., Bulletin 12, Min. Ind. Exp. Sta., Pennsylvania State College, 1933.

[3] Muskat, M., Flow of Homogeneous Fluids, McGraw-Hill, New York, 1937.

and so

$$\left.\begin{aligned} v_x &= K\frac{\partial \phi}{\partial x} \\ v_y &= K\frac{\partial \phi}{\partial y} \\ v_z &= K\frac{\partial \phi}{\partial z} \end{aligned}\right] \quad \ldots\ldots\ldots\ldots (9)$$

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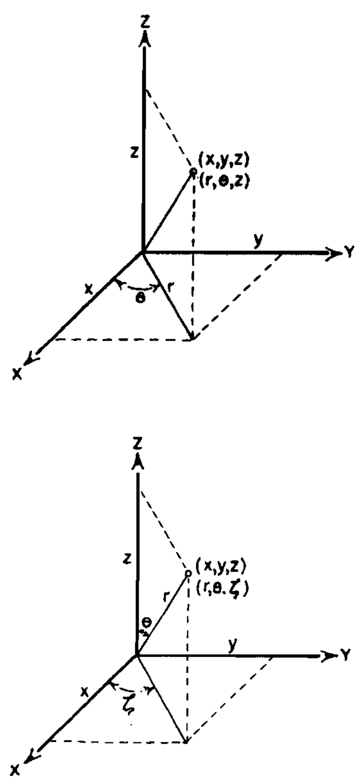


b. Spherical coordinates (r,θ,ζ).

Figure 1 - Two Coordinate Systems.

and equation (11) now becomes

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \ldots(13)$$

If the flow is not a function of θ, equation (13) may be written

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \ldots(14)$$

For spherical coordinates (r, θ, ζ) (see figure 1b), the velocity components become

$$\left.\begin{aligned} v_r &= \frac{\partial\phi}{\partial r} \\ v_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} \\ v_\zeta &= \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\theta} \end{aligned}\right] \quad \ldots.(15)$$

and equation (11) now becomes

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\zeta^2} = 0 \ldots(16)$$

A function can be obtained that defines the path along which a fluid particle moves in traveling through a soil mass. This function is called a stream function and is given the Greek letter ψ. It is related to the potential function ϕ through the equations

$$\left.\begin{aligned} \frac{\partial\psi}{\partial y} &= \frac{\partial\phi}{\partial x} \\ -\frac{\partial\psi}{\partial x} &= \frac{\partial\phi}{\partial y} \end{aligned}\right] \quad \ldots\ldots..(17)$$

The velocities then become, in terms of ψ,

$$\left.\begin{aligned} v_x &= K\frac{\partial\psi}{\partial y} \\ v_y &= -K\frac{\partial\psi}{\partial x} \end{aligned}\right] \quad \ldots..(18)$$

ANALYTICAL SOLUTIONS (STEADY STATE)

General. In the preceding section it has been shown that Darcy's law applies to problems in steady-state slow flow through porous media, and also what conditions the potential function must satisfy in order to offer solutions to flow problems. Flow problems may be solved by analytical or experimental means or a combination of the two. A few analytical solutions are presented on the following pages.

Two-Dimensional Radial Flow (Point-Source or Sink). Here we assume a two-dimensional flow system (figure 2) in which the velocities vary only with the distance, r, from the point-source (line-source from a three-dimensional point of view), so that

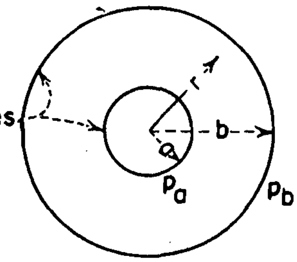


Figure 2 - Two-Dimensional Radial Flow.

$$v_r = K\frac{\partial p}{\partial r}$$

and

$$v_\theta = \frac{K}{r}\frac{\partial p}{\partial\theta} = 0$$

where

v = velocity,
K = permeability coefficient, and
p = pressure head,

and cylindrical coordinates are used.

Equation (6) may now be written

$$\nabla^2 p = \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} = 0 \quad ..(19)$$

The function

$$p = C_1 \ln r + C_2 \quad \ldots\ldots(20)$$

will be found to satisfy equation (19). Then we have, for the boundary conditions,

$$\left.\begin{aligned} r &= a, p = p_a \\ r &= b, p = p_b \end{aligned}\right] \quad \ldots\ldots\ldots\ldots (21)$$

where a and b are respective radii from the point-source to two arbitrary equipotential lines (see figure 2).

Using conditions (21) successively in equation (20),

$$p_a = C_1 \ln a + C_2$$

$$p_b = C_1 \ln b + C_2$$

and

$$C_1 = \frac{p_b - p_a}{\ln \frac{b}{a}}$$

$$C_2 = \frac{p_a \ln b - p_b \ln a}{\ln \frac{b}{a}}$$

Placing the values of C_1 and C_2 into equation (20) gives

$$p = \frac{p_b - p_a}{\ln \frac{b}{a}} \ln \frac{r}{a} + p_a \quad \ldots (22)$$

Then, by differentiation,

$$v_r = K \frac{\partial p}{\partial r} = \frac{K}{r} \frac{(p_b - p_a)}{\ln \frac{b}{a}} \quad . \ (23)$$

The total flow, Q, is given by the equation

$$Q = z \int_0^{2\pi} r\, v_r\, d\theta$$

$$= \frac{2\pi K z\ (p_b - p_a)}{\ln \frac{b}{a}} \quad \ldots (24)$$

where z is the thickness of the porous medium. Equations (22) and (23) may be written in terms of Q as follows:

$$p = \frac{Q}{2\pi K z} \ln \frac{r}{a} + p_a \quad \ldots\ldots (25)$$

$$v_r = \frac{Q}{2\pi r z} \quad \ldots\ldots\ldots\ldots (26)$$

Example 1. In foundation tests for Deer Creek Dam, Provo River Project, Utah, a 12-inch diameter well was drilled 84 feet to bedrock, and twenty observation wells were located symmetrically on 5- to 200-foot radii from the 12-inch well. After pumping 210 gallons per minute (0.4679 second-feet) for 87 hours a steady state was approached and the following data were observed. (The two equipotential lines in figure 2 were arbitrarily taken at distances of 10 and 200 feet, respectively.)

At 200-foot radius, average water elevation = 5276.5
At 10-foot radius, average water elevation = 5274.6
z = thickness of bed at 10-foot radius = 78.9 feet.

Inserting these values in equation (24) and solving for K,

$$K = \frac{(0.4679)\ \ln \frac{200}{10}}{2\pi\ (78.9)(5276.5 - 5274.6)}$$

$$= 0.0015 \text{ feet per second.}$$

Two-Dimensional Flow Between an Infinite Line-Source and a Point-Sink. (Method of Images.) This is similar to the problem of radial flow into a well except that in this case a linear source rather than a circular source represents the external boundary (see figure 3). This solution also is two dimensional.

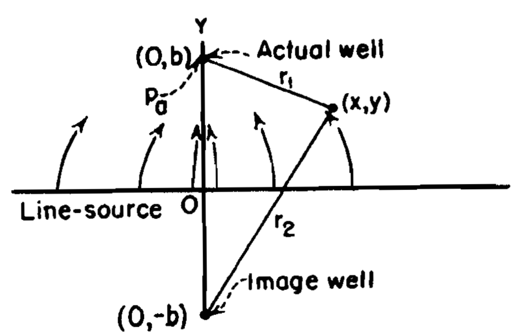


Figure 3 - Flow between a Line-Source and a Point-Sink.

In this development it will be assumed that an infinite line-source extends along the X axis and that at a distance b from the X axis is a well of radius $r = a$. For the moment, it will also be assumed that the pressure along the line-source is maintained at zero and that the well pressure is p_a. In two-dimensional problems, one may represent any well with uniform pressure, p_a, at the periphery by a point-source or sink at the center of the well. The potential function for this case will then become

$$p = C \ln \frac{r}{a} + p_a \quad \ldots\ldots\ldots (27)$$

where C is a constant determined by the boundary conditions. Since there is no sand or porous medium in the well, equation (27) does not hold for the well interior (r less than a).

If there are several wells in a system, each well will contribute an amount p of equation (27) to the resultant pressure distribution. The r to any point, of course, must in all cases be measured from the centers of the individual wells.

It will be noted that the streamlines from the line-source, or boundary, have the same pattern and direction as though they had come from a point-source or well at the point (0, - b). In this discussion, this well will be called an image well, and, in particular, a negative image well. It will be substituted for the line-source to facilitate evaluation of the pressure distribution. If we add the pressure contributions of the actual well and the image well, for any point (x, y), the resultant effect will be

$$p = C \ln \frac{r_1}{a} + p_a - C \ln \frac{r_2}{a} - p_a$$

or.

$$p = C \ln \frac{r_1}{r_2} \quad \ldots\ldots\ldots\ldots (28)$$

where r_1 and r_2 are the distances to any point from the centers of the actual and image wells, respectively. It is to be noted that when $r_1 = r_2$, $p = 0$, the condition assumed along the X axis.

We shall now proceed to find the value of p over the actual well surface. It is assumed that the radius, a, of the well is small compared with 2b, the distance between the actual well and the image well. Over the actual well periphery, then, $r_1 = a$ and $r_2 = 2\ b$, so that equation (28) becomes

$$p_a = C \ln \frac{a}{2\ b}$$

from which

$$C = \frac{p_a}{\ln \frac{a}{2\ b}} \quad \ldots\ldots\ldots\ldots (29)$$

Substituting this value of C into equation (28) gives

$$p = \frac{p_a}{\ln \frac{a}{2\ b}} \ln \frac{r_1}{r_2} \quad \ldots\ldots\ldots (30)$$

It may be more convenient in special cases to specify the pressure over the line-source as p_o rather than zero, where p_o may have any value. In this case, then,

$$p = \frac{p_a - p_o}{\ln \frac{a}{2\ b}} \ln \frac{r_1}{r_2} + p_o$$

$$= \frac{1}{2} \frac{p_o - p_a}{\ln \frac{2\ b}{a}} \ln \frac{x^2 + (y - b)^2}{x^2 + (y + b)^2}$$

$$+ p_o \quad \ldots\ldots\ldots\ldots\ldots (31)$$

The total flow from the line-source to the well is given by the equation

$$Q = K z \int_{-\infty}^{+\infty} \left[\frac{\partial p}{\partial y}\right]_{y = o} . \ dx$$

$$= \frac{2 K b (p_o - p_a) z}{\ln \frac{2\ b}{a}} \int_{-\infty}^{+\infty} \frac{dx}{x^2 + b^2}$$

or

$$Q = \frac{2 \pi K z (p_o - p_a)}{\ln \frac{2\ b}{a}} \quad \ldots\ldots (32)$$

Example 2. In the foundation test for Deer Creek Dam, Provo River Project, Utah, water elevations in observation wells showed that the direction of drainage was from the ground to the river. If the groundwater table had been lower so that seepage from the river alone supplied the well, and the steady-state discharge of 210 gallons per minute had caused the observation-well drawdown used in Example 1, the K could have been determined from equation (32). Then, using an infinite line-source 200 feet from the well to represent the river, at Elevation 5276.5, and an average groundwater elevation of 5274.6 at 10-foot radius from the well, and solving equation (32) for K, we would, after substitution, have

$$K = \frac{(0.4679) \ln \frac{2\,(200)}{10}}{2\,\pi\,(78.9)(5276.5 - 5274.6)}$$

$$= 0.0018 \text{ feet per second.}$$

Thus, if this type of flow had existed, the data would have shown 20 percent higher permeability than in Example 1. Conversely, if the K of 0.0015 computed in Example 1 had been retained, and the groundwater level had been lowered so that equation (32) applied, then 17 percent less well discharge would have caused the drawdown noted.

Two-Dimensional Finite Line-Source or Sink Applied to Flow Beneath Impervious Dam on a Pervious Foundation. It has been shown and is generally known that the streamlines under an impervious dam resting on a pervious infinite foundation are a system of confocal ellipses with center at 0 (see figure 4). The base of the dam, AB, is a streamline and is the limiting form of this family of ellipses.

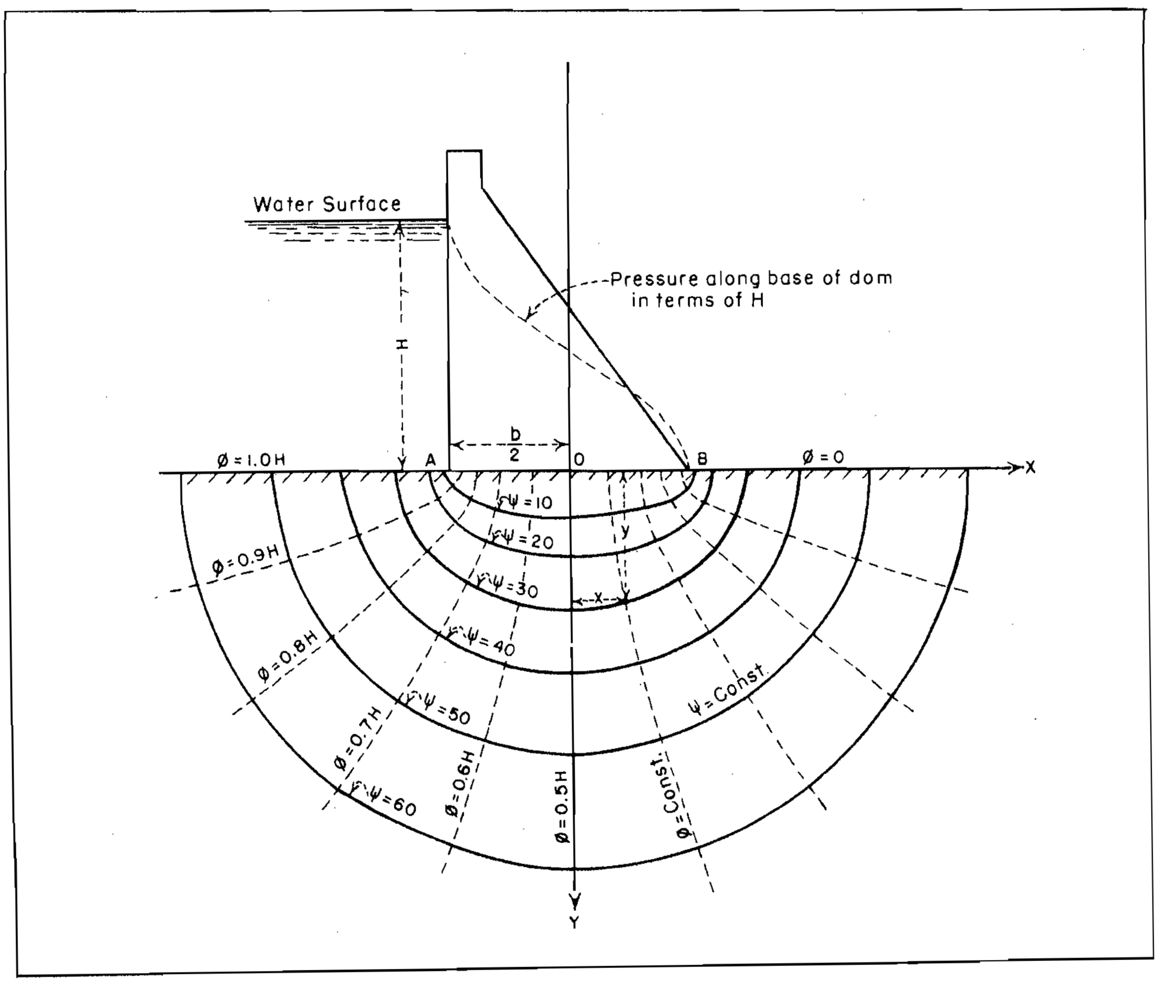


Figure 4 - Flow beneath an Impervious Dam on a Pervious Foundation.

The stream function, ψ (which represents physically the total flow crossing any line connecting the origin with a point (x, y) in the flow system) and the potential function, ϕ, passing through the same point, can be expressed by the complex function

$$z = \frac{b}{2} \cosh \frac{\pi}{H} w \quad \dots\dots\dots \quad (33)$$

in which

$$z = x + i\,y$$

$$w = \psi + i\,\phi$$

Placing the expressions for z and w in equation (33) gives

$$x + i\,y = \frac{b}{2} \cosh \left(\frac{\pi}{H}\psi + i\frac{\pi}{H}\phi\right)$$

$$x + i\,y = \frac{b}{2} \cosh \frac{\pi}{H}\psi \cos \frac{\pi}{H}\phi + i \sinh \frac{\pi}{H}\psi \sin \frac{\pi}{H}\phi$$

On equating the real and imaginary parts, we get

$$x = \frac{b}{2} \cosh \frac{\pi}{H}\psi \cos \frac{\pi}{H}\phi \quad .(34)$$

$$y = \frac{b}{2} \sinh \frac{\pi}{H}\psi \sin \frac{\pi}{H}\phi \quad ..(35)$$

Solving for $\cos \frac{\pi}{H}\phi$ and $\sin \frac{\pi}{H}\phi$,

$$\cos \frac{\pi}{H}\phi = \frac{x}{\frac{b}{2}\cosh \frac{\pi}{H}\phi} \quad \dots (36)$$

$$\sin \frac{\pi}{H}\phi = \frac{y}{\frac{b}{2}\sinh \frac{\pi}{H}\psi} \quad \dots (37)$$

Then squaring and adding,

$$\cos^2 \frac{\pi}{H}\phi + \sin^2 \frac{\pi}{H}\phi = 1$$

$$= \frac{x^2}{\left(\frac{b}{2}\cosh \frac{\pi}{H}\psi\right)^2} + \frac{y^2}{\left(\frac{b}{2}\sinh \frac{\pi}{H}\psi\right)^2} \quad (38)$$

Similarly, solving for $\cosh \frac{\pi}{H}\psi$ and $\sinh \frac{\pi}{H}\psi$, we have

$$\cosh^2 \frac{\pi}{H}\psi - \sinh^2 \frac{\pi}{H}\psi = 1$$

$$= \frac{x^2}{\left(\frac{b}{2}\cos \frac{\pi}{H}\phi\right)^2} - \frac{y^2}{\left(\frac{b}{2}\sin \frac{\pi}{H}\phi\right)^2} \quad .(39)$$

The uplift pressures along the base of the dam ($y = 0$) may be found from equation (34) by substitution of $\psi = 0$. Then equation (34) becomes

$$x = \frac{b}{2} \cos \frac{\pi}{H}\phi \quad \dots\dots\dots \quad (40)$$

and

$$p = \phi = \frac{H}{\pi} \cos^{-1} \frac{2x}{b} \quad \dots(41)$$

Note that the boundary conditions are satisfied in equation (41), with

$$\phi = H \text{ at } x = -\frac{b}{2}$$

and

$$\phi = 0 \text{ at } x = +\frac{b}{2}$$

The uplift pressures along the base of the dam are given in figure 4 in terms of H, the acting head.

Streamlines and equipotential lines may be plotted by the use of equations (38) and (39), respectively. Then with the flow net established, the seepage losses, Q, may be determined by use of the relations

$$\left.\begin{aligned} v &= K\frac{d\phi}{ds} \\ Q &= A\,v \end{aligned}\right] \quad \dots\dots\dots\dots \quad (42)$$

in which

K = the permeability coefficient,
$\frac{d\phi}{ds}$ = pressure-head gradient (dimensionless),
v = velocity,
A = cross-sectional area, and
Q = discharge or seepage loss.

Two-Dimensional Finite Line-Source or Sink Applied to Flow Around Sheet-Piling[4]. Consider the function

$$z = b \cosh w \quad \ldots\ldots\ldots\ldots (43)$$

Again

$$z = x + i\,y$$

$$w = \psi + i\,\phi$$

Placing these values in equation (43) gives

$$x + i\,y = b \cosh(\psi + i\,\phi)$$

$$x + i\,y = b \cosh\psi \cos\phi + i\,b \sinh\psi \sin\phi$$

On equating the real and the imaginary parts, we get

$$x = b \cosh\psi \cos\phi \quad \ldots\ldots\ldots (44)$$

$$y = b \sinh\psi \sin\phi \quad \ldots\ldots\ldots (45)$$

Solving for $\cos\phi$ and $\sin\phi$, we get

$$\cos\phi = \frac{x}{b \cosh\psi} \quad \ldots\ldots\ldots (46)$$

$$\sin\phi = \frac{y}{b \sinh\psi} \quad \ldots\ldots\ldots (47)$$

Then squaring and adding,

$$\frac{x^2}{b^2 \cosh^2\psi} + \frac{y^2}{b^2 \sinh^2\psi} = 1 \quad . (48)$$

Similarly, solving for $\cosh\psi$ and $\sinh\psi$, we have

$$\frac{x^2}{b^2 \cos^2\phi} - \frac{y^2}{b^2 \sin^2\phi} = 1 \quad . . (49)$$

From equations (44) and (45) it can be seen that $\phi = \pi$ represents the negative part of the X axis beyond $x = -b$. If equation (49) is written

$$\frac{x^2}{b^2} - \frac{y^2 \cos^2\phi}{b^2 \sin^2\phi} = \cos^2\phi$$

it can be seen that $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$ represent the line $x = 0$, or the Y axis.

Now if the coordinate system is drawn with the X axis positive upward and the Y axis positive to the left, the function given in equation (43) is seen to represent the flow around a sheet-piling wall of depth b, shown in figure 5. To correctly represent the flow, the branches of the equipotential lines falling in the second quadrant only should be drawn for values of ϕ between π and $3\pi/2$, and the branches falling in the third quadrant only should be drawn for values of ϕ between $\pi/2$ and π.

Note in this development that the upstream potential is given as $\phi = 3\pi/2$ and the downstream potential as $\phi = \pi/2$. This means that the head of water upstream must correspond to $3\pi/2$ and the head downstream must correspond to $\pi/2$. (See figure 5.) This can better be seen by considering the potential function with g/γ equal to unity, or

$$\phi = p \pm y \quad \ldots\ldots\ldots\ldots (50)$$

It is known that on the upstream foundation line, $p = H_1 + H_2$, and $y = 0$; hence, by equation (50), $\phi = H_1 + H_2$. The mathematical solution gives $\phi = 3\pi/2$, so $H_1 + H_2 = 3\pi/2$. Similarly, along the downstream foundation line it is known that $p = H_2$ and $y = 0$, hence $\phi = H_2$. The mathematical solution gives $\phi = \pi/2$ along this line, so $H_2 = \pi/2$. The results of this study are shown in figure 5.

Three-Dimensional Radial Flow (Point-Source or Sink). Many problems in the flow of fluids through porous media can be adapted to a two-dimensional flow system, but occasionally there are problems that can be treated only by a three-dimensional solution.

In three-dimensional problems it becomes necessary to consider gravity. The potential function given as equation (8) is

$$\phi = p + \frac{g_z\, z}{\gamma}$$

and Laplace's equation, equation (11), in terms of ϕ, is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

[4] Vetter, C. P., Notes on Hydrodynamics (Volume I), Technical Memorandum No. 620, U. S. Bureau of Reclamation, Denver, Colorado, September 1941, p. 100.

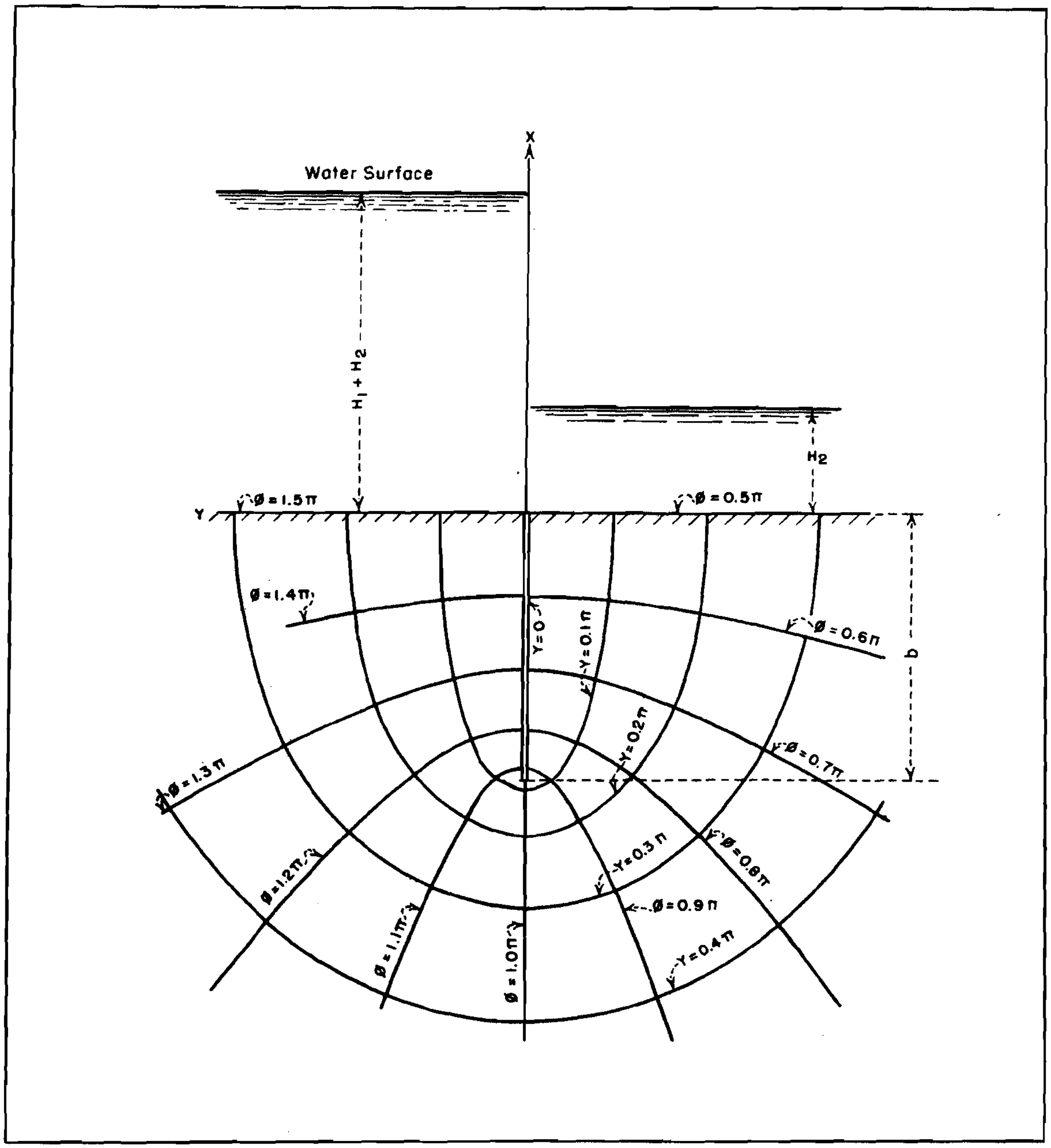


Figure 5 - Flow around Sheet-piling.

Spherical Flow. Spherical flow is analogous to the two-dimensional problem of radial flow, for here the potential and velocity distributions will depend only on the radius, r, of a spherical coordinate system. Laplace's equation in a spherical coordinate system (r, θ, ζ) will have the general form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \zeta^2} = 0 \quad . . . (51)$$

but in the case of spherical flow this reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) = 0 \ldots\ldots (52)$$

It will be found that the potential function,

$$\phi = -\frac{C_1}{r} + C_2 \ldots\ldots\ldots (53)$$

is a solution to equation (52) and that it gives the potential ϕ throughout a spherical flow system. The constants C_1 and C_2 can be determined from the boundary conditions. In figure 6

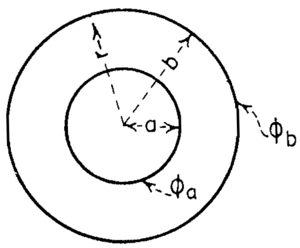


Figure 6 - Three-Dimensional Radial Flow.

$$\left.\begin{array}{l}\phi = \phi_a \text{ at } r = a\\ \phi = \phi_b \text{ at } r = b\end{array}\right] \ldots\ldots\ldots (54)$$

Substitution of equation (54) in equation (53) gives

$$\phi_a = -\frac{C_1}{a} + C_2$$

$$\phi_b = -\frac{C_1}{b} + C_2$$

$$C_1 = \frac{\phi_b - \phi_a}{\frac{1}{a} - \frac{1}{b}}$$

$$C_2 = \phi_a + \frac{(\phi_b - \phi_a)}{\frac{1}{a} - \frac{1}{b}}\,\frac{1}{a}$$

$$\phi = \frac{\phi_b - \phi_a}{\frac{1}{b} - \frac{1}{a}}\left[\frac{1}{r} - \frac{1}{a}\right] + \phi_a \ldots (55)$$

Then the velocity is

$$v_r = K\frac{\partial \phi}{\partial r}$$

$$= -\frac{(\phi_b - \phi_a)}{\frac{1}{b} - \frac{1}{a}}\,\frac{K}{r^2} \ldots\ldots (56)$$

The total flow through the system is given by

$$Q = \int_0^{2\pi} dx \int_0^{\pi} r^2 \sin\theta\, v_r\, d\theta$$

$$= \frac{4\pi(\phi_b - \phi_a)K}{\frac{1}{a} - \frac{1}{b}} \ldots\ldots (57)$$

Using equation (57) with equations (55) and (56),

$$\phi = \frac{Q}{4\pi K}\left[\frac{1}{a} - \frac{1}{r}\right] + \phi_a \ldots (58)$$

and

$$v_r = \frac{Q}{4\pi r^2} \ldots\ldots\ldots\ldots\ldots (59)$$

Example 3. Equation (57) may be used in many cases for the determination of the average coefficient of permeability of a soil by means of a simple field experiment. At the Elk Creek Damsite, Conejos River, San Luis Valley Project, a hole was drilled into the soil foundation and an open end casing with an inside diameter of 5.75 inches was sunk into the bed. All earth material was cleaned out of the casing down to the level of the bottom. A measured flow of water was supplied to the casing, the inside diameter of the pipe was noted, and also the head differential between the water level inside and outside of the casing. In applying equation (57) to this problem, it is assumed that hemispherical flow takes place, and that the outer radius of the sphere, b, becomes infinite. Then with $\phi_b - \phi_a$ equal to H, or the head differential inside and outside of the casing, equation (57) becomes

$$Q = \frac{2\pi H K}{\frac{1}{a}}$$

or

$$K = \frac{Q}{2\pi a H} \ldots\ldots\ldots\ldots (60)$$

Data received from tests at the Elk Creek Damsite showed that at Drillhole No. 3, with the open end of the casing 25.0 feet below the ground surface, H = 8.8 feet, Q = 0.006, 996 second-feet, and a = 2.875/12 = 0.240

feet. Then

$$K = \frac{0.006,996}{2\pi \times 0.240 \times 8.8}$$

$$= 527.2 \times 10^{-6} \text{ feet per second}$$

$$= 16,630 \text{ feet per year.}$$

A second test made with the open end of the casing 47.0 feet below the ground surface gave $H = 9.8$ feet, $Q = 0.001,493$ second-feet, and $a = 0.240$ feet. Then in this instance

$$K = \frac{0.001,493}{2\pi \times 0.240 \times 9.8}$$

$$= 101.2 \times 10^{-6} \text{ feet per second}$$

$$= 3,190 \text{ feet per year.}$$

Electric analogy experiments show that when using a casing with a flat bottom for the field determination of K in the manner described above, the equation should be

$$K = \frac{Q}{5.553\, a\, H} \quad \ldots\ldots\ldots \quad (61)$$

which differs from equation (60) in that the constant in the denominator is 5.553 rather than 2π. The difference results from the fact that the flow is not truly hemispherical. Equation (61) will give K-values 13 percent greater than equation (60) for the same field data.

Analytical Determination of Critical Exit Gradients. Water, in percolating through a soil mass, has a certain residual force at each point along its path of flow and in the direction of flow which is proportional to the pressure gradient at that point. When the water emerges from the subsoil, this force acts in an upward direction and tends to lift the soil particles. Once the surface particles are disturbed, the resistance against the upward pressure of the percolating water is further reduced, tending to give progressive disruption of the subsoil. The flow in this case tends to form into "pipes," and it is this concept that has brought about the commonly accepted term of "piping." This action may also be described as a flotation process in which the pressure upward exceeds the downward weight of the soil mass. Since the soil is saturated, it is apparent that the upward pressure gradient, F, of the percolating water must be equal to the wet density, W, of the soil in order to produce the critical or flotation gradient. The statement may be proved as follows:

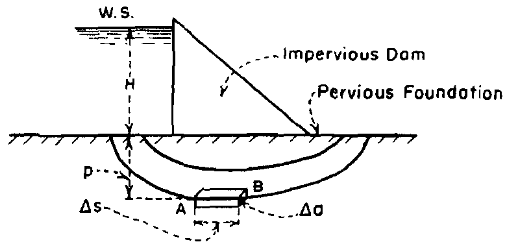


Figure 7 - Element of Soil in Pervious Foundation.

Consider the rectangular parallelepiped shown at the bottom of figure 7, bounded by streamlines and equipotential lines, with end area Δa and length Δs. The force at face A is equal to $p\,\Delta a$, and at face B is equal to $(p + \Delta p)\,\Delta a$. Then neglecting the curvature of the streamline, the net force acting on the element in the direction of the streamline is given by the equation

$$p\,\Delta a - (p + \Delta p)\,\Delta a = -\Delta p\,\Delta a \;..(62)$$

and since the volume of the element is $\Delta s \Delta a$, the force per unit of volume becomes

$$F = -\frac{\Delta p \Delta a}{\Delta s \Delta a} = -\frac{\Delta p}{\Delta s}$$

If Δs is made to approach zero this gives

$$F = -\frac{dp}{ds} \quad \ldots\ldots\ldots\ldots\ldots \quad (63)$$

in which dp/ds is the pressure gradient at the point.

Now consider figure 8. At each point along a streamline the two forces, W and F, will be acting. Their effects can be resolved into a resultant, R, at that point.

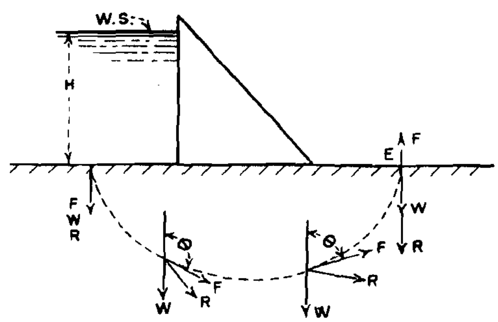


Figure 8 - Force Components.

For stability there must be no upward component of the resultant. The vertical component of R is

$$R_v = W - F \cos \theta \quad \ldots\ldots\ldots \quad (64)$$

The dangerous region in a structure is near the point, E, where $\theta = 0^o$. Soil particles at E will be on the verge of failure if $R = 0$. This condition will define the limiting case, or

$$F = W \quad \ldots\ldots\ldots\ldots\ldots \quad (65)$$

But equation (63) shows that F was the pressure gradient at any point. So the critical gradient becomes

$$F = -\frac{dp}{ds} = W \quad \ldots\ldots\ldots \quad (66)$$

The above equation states that the critical or flotation gradient is equal to the wet density of the soil.

A mathematical method[5] has been developed for determining the critical gradients, or in particular the exit gradients in the vicinity of a cut-off wall extending into the pervious foundation material. The method is based upon a function of the complex variable and makes use of the Schwarz-Christoffel transformation. The derivation is not given here because of space limitations, but the results that follow give the findings for the hydraulic exit gradient, G_E, at the critical point for several cases.

Case One. This is for a single pile-line with no step, and no apron upstream or downstream.

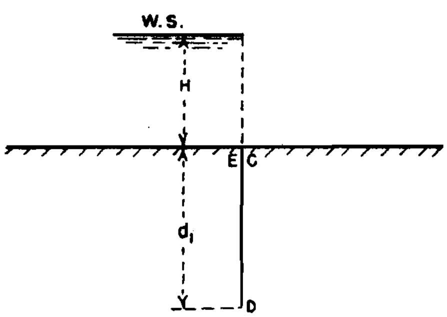


Figure 9 - Exit Gradients: Case One.

[5] Khosla, A. N., Bose, N. K., and Taylor, E. M., Design of Weirs on Permeable Foundations, Publication 12, Central Board of Irrigation, India, September, 1936.

At Point C

$$G_E = \frac{H}{\pi d_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (67)$$

Case Two. This is for a single pile-line with step, and no apron upstream or downstream.

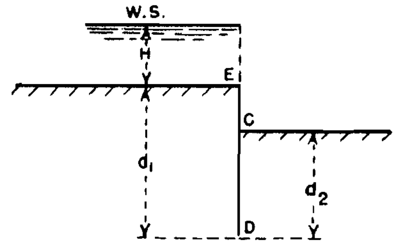


Figure 10 - Exit Gradients: Case Two.

At Point C

$$G_E = \frac{H}{d_2} \cdot \frac{d_2}{d_1 - d_2} \cdot \frac{c}{1 - c} \quad \ldots \quad (68)$$

where

$$c = \cos \theta$$

and

$$\tan \theta - \theta = \pi \frac{d_2}{(d_1 - d_2)}$$

The meaning of all symbols used is evident from the sketches given above with the exception of c, which is a function involving d_1, d_2, and π, and is most readily determined from the curves of figures 11, 12, and 13.

The value of G_E is obtained either by calculation or by interpolation in Table 1.

VALUES OF G_E (CASE TWO)

(After Khosla, et al, Publication 12, Central Board of Irrigation, India.)

c	$\frac{d_2}{d_1 - d_2}$	$G_{exit} \div \frac{H}{d_2}$
1.000	0	-
0.645	0.1	0.182
0.516	0.2	0.213
0.437	0.3	0.233
0.380	0.4	0.245
0.335	0.5	0.252
0.302	0.6	0.260
0.275	0.7	0.265
0.254	0.8	0.270
0.233	0.9	0.274
0.217	1.0	0.278
0.129	2.0	0.295
0.091	3.0	0.301
0.071	4.0	0.305
0.058	5.0	0.307
0.049	6.0	0.310
0.042	7.0	0.310
0.038	8.0	0.311
0.033	9.0	0.312
0.030	10.0	0.312

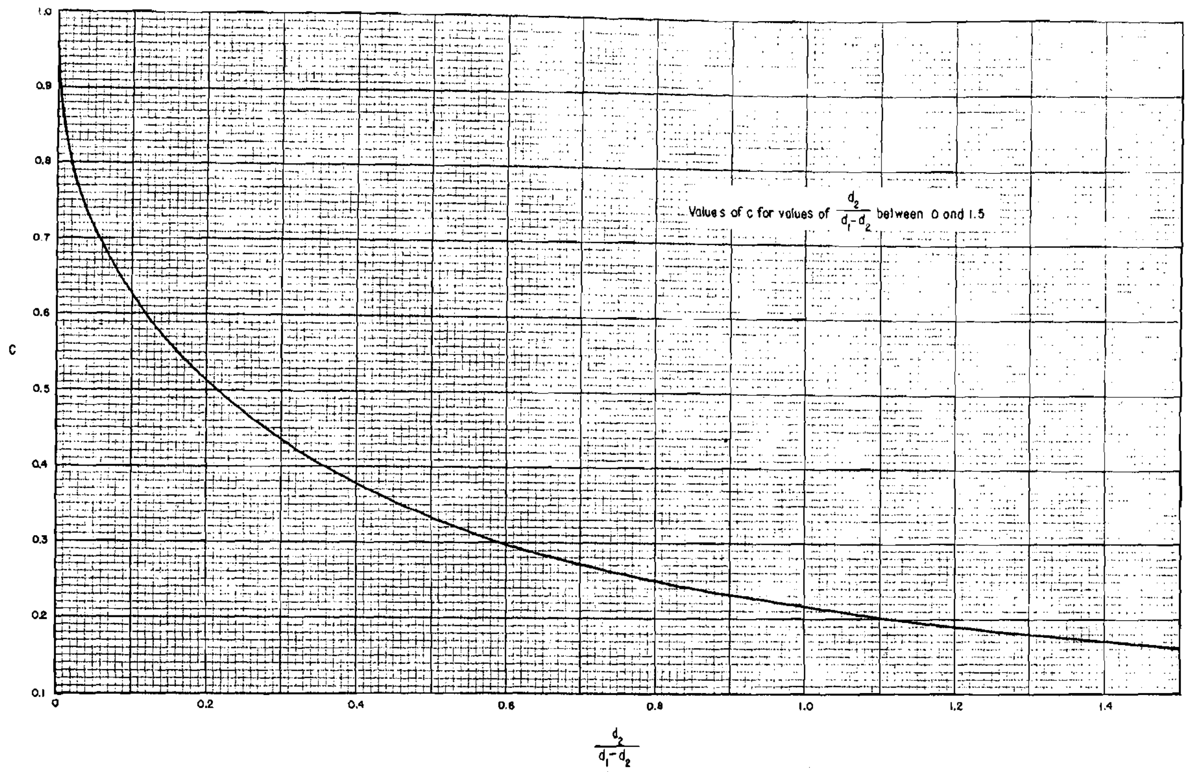


Figure 11 - Values of c for Use in Determining Exit Gradient.

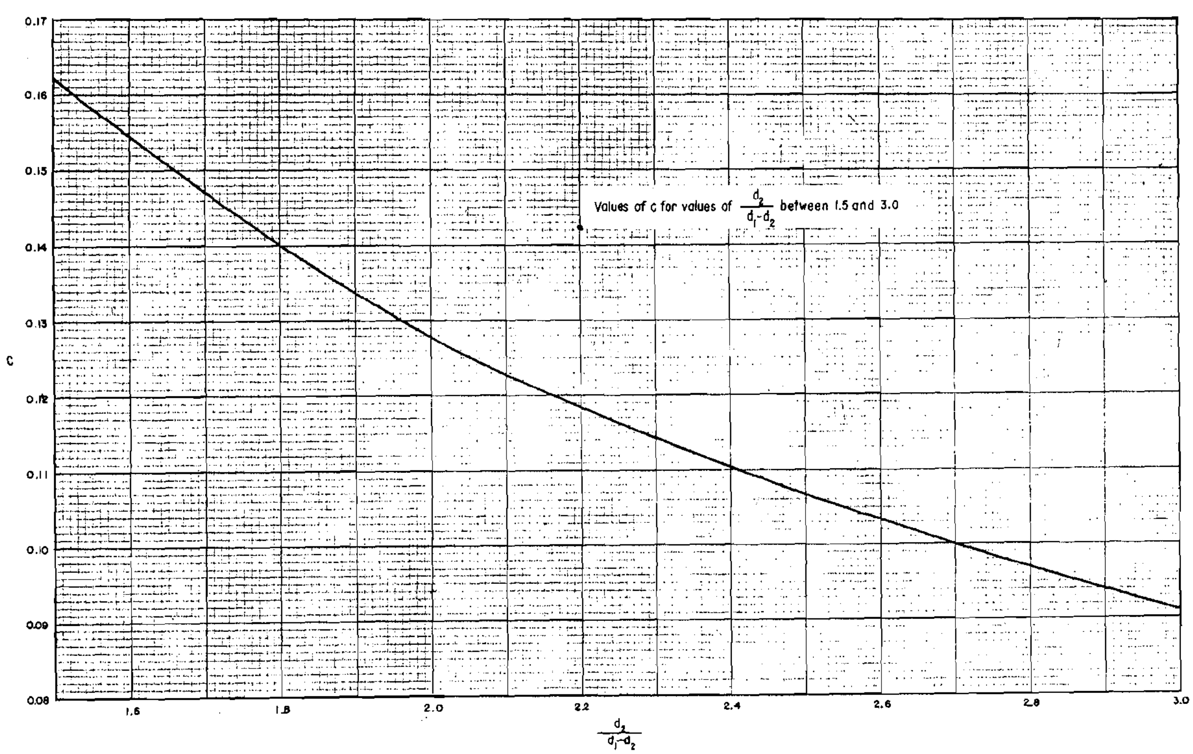


Figure 12 - Values of c for Use in Determining Exit Gradient (Continued).

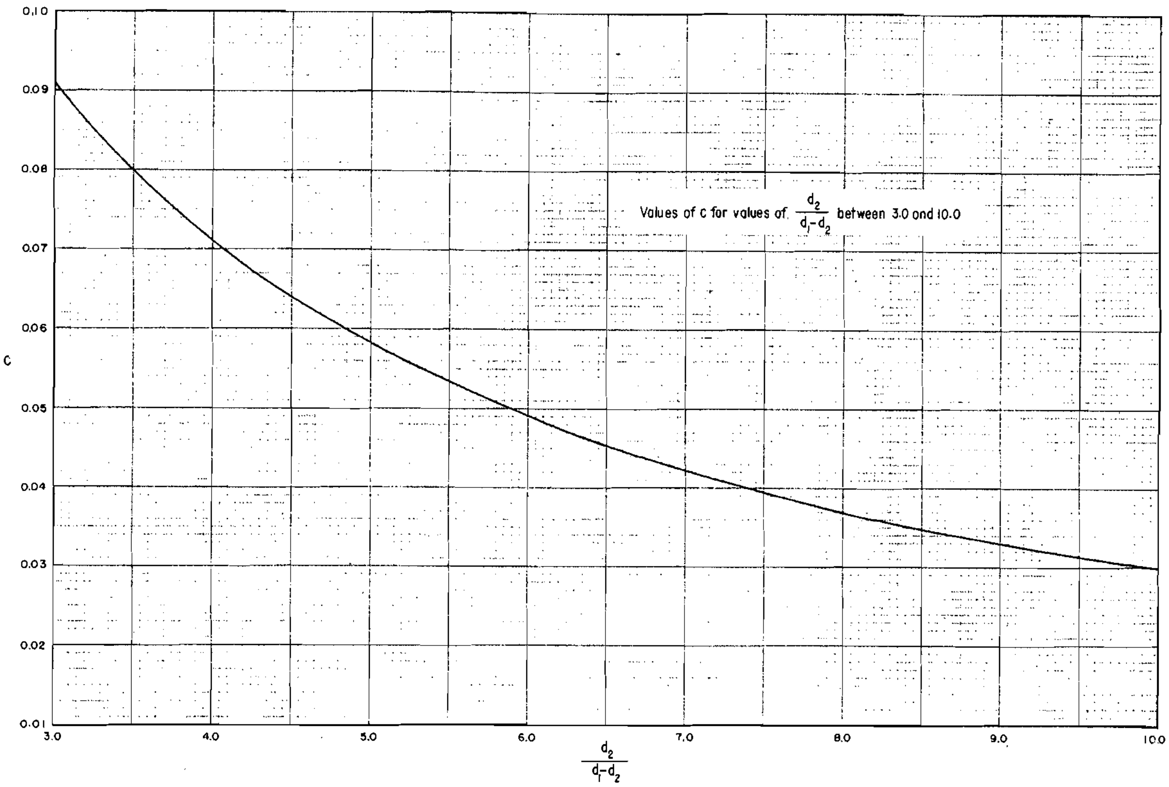


Figure 13 - Values of c for Use in Determining Exit Gradient (Concluded).

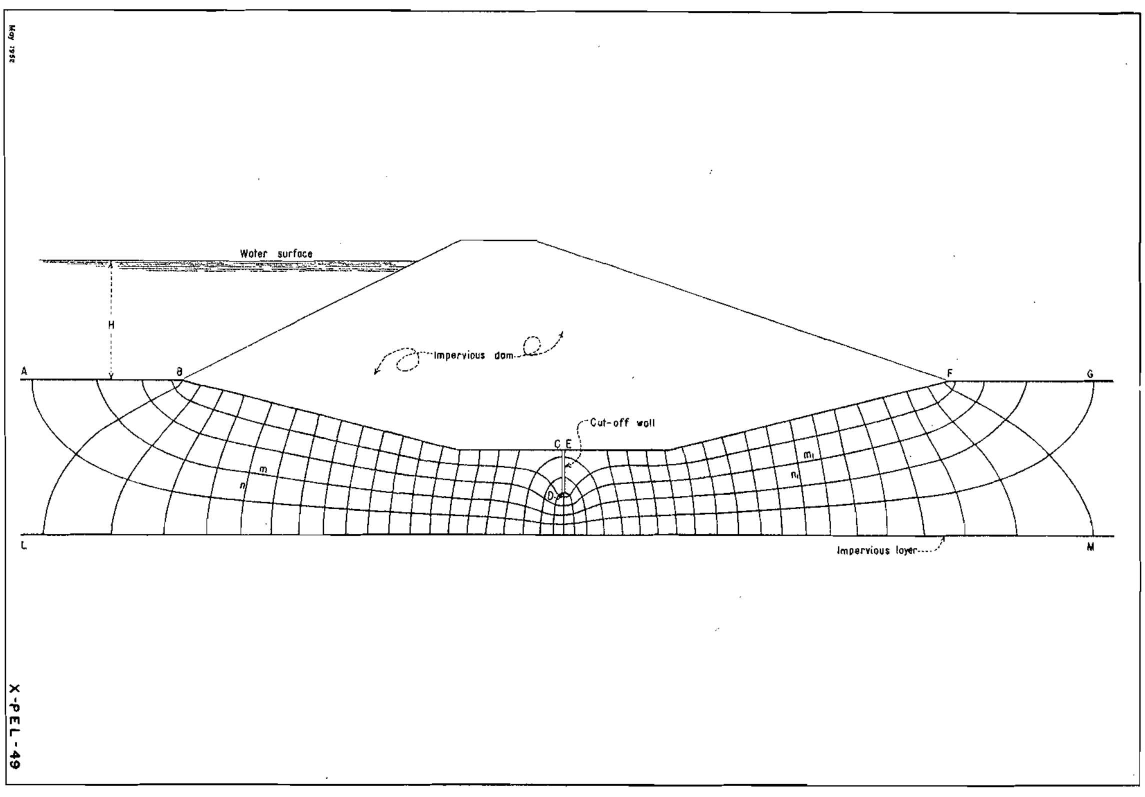


Figure 14 - Forchheimer's Graphical Solution of an Impervious Dam on a Pervious Foundation.

Case Three. This is for a dam with no pile-line.

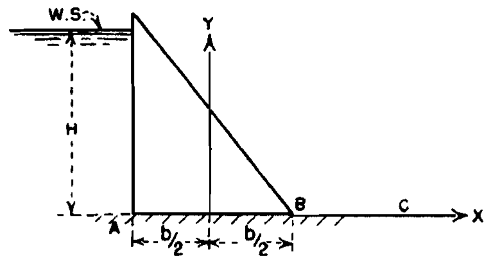


Figure 15 - Exit Gradients: Case Three.

At Point B

$$G_E = \text{infinity}$$

Along BC

$$G_E = \frac{2H}{\pi b} \frac{1}{\sqrt{\left[\frac{2x}{b}\right]^2 - 1}} \quad \text{. . (69)}$$

Example 4. Equations (67) and (68) may also be used to determine the depth of piles to give a desired exit gradient. First consider the problem as shown in figure 9.

Assume H = 14 feet and that it is desired to have an exit gradient of 0.2. (This will give a safety factor of 5.) From equation (67),

$$d_1 = \frac{H}{\pi G_E} = \frac{14}{\pi (0.2)}$$

$$= 22.3 \text{ feet.}$$

Now assume the problem to be as shown in figure 10. From equation (68) with $G_E = 0.2$ and H = 14 feet, and letting $d_1 - d_2 = 14$ feet,

$$G_E = \frac{14}{d_2} \cdot \frac{d_2}{14} \cdot \frac{c}{1 - c} = 0.2$$

or

$$\frac{c}{1 - c} = 0.2, \quad c = 0.167$$

By definition,

$$c = \cos \theta = 0.167$$

$$\theta = 1.403$$

and by definition

$$\tan \theta - \theta = \frac{\pi d_2}{d_1 - d_2}$$

Therefore

$$\frac{\pi}{14} d_2 = 4.501$$

or

$$d_2 = 20.0 \text{ feet.}$$

Factor of Safety. Theoretically, a structure would be safe against piping if the exit gradients are only slightly smaller than the wet density of the soil. However, there are many factors, such as washing of the surface and earthquake effects, that could easily change such a stable condition into one of incipient failure. For this reason it is desirable to have a factor of safety so that the exit gradients are much smaller than the critical value. Although the question of amount of factor of safety has not been settled, values of 4.0 to 6.0 have been proposed, the smaller value for coarse material and the larger value for fine sand.

EXPERIMENTAL SOLUTIONS

General. By choosing the appropriate analytical solutions to the Laplace equation and combining their effects, many flow problems can be solved. Solutions other than those previously mentioned, however, are usually cumbersome, and the use of experimental methods is justified. The five commonly used experimental methods are:

1. Graphical construction of flownets.
2. Membrane analogy.
3. Electric analogy.
4. Hydraulic models (including Viscous-Fluid Method).
5. Field experiments on the actual structure.

Each of these methods has an advantageous field of application. Of the five methods, the electric analogy has, in most cases, been found to give the best accuracy with the least cost and greatest speed. Where transitory effects are of major interest, the use of hydraulic scale models is justified. The first four methods will be treated separately in the following paragraphs.

Graphical Construction of Flow Nets. It is known that streamlines and equipotential lines are everywhere normal to each other. Ignoring the effect of gravity, all boundaries of a flow system must also be either streamlines or equipotential lines. It is possible then to make a sketch of a flow system, starting with the known boundary conditions. Professor Forchheimer[6] introduced this method some forty years ago. The method is approximate, but gives results which are generally sufficiently accurate for practical purposes.

The method can be best demonstrated by considering the sketch shown in figure 14. It is assumed here that an impervious dam rests upon a pervious layer of foundation material which in turn rests upon an impervious rock foundation. A cut-off wall at the center of the dam extends approximately halfway into the pervious material. The horizontal upstream line, AB, is a line of equipotential as is also the line FG. They are, however, not at the same potential, but differ in potential by the depth of water in the reservoir, H. The line BCDEF and the line LM are immediately known to be streamlines. Therefore, it is only necessary to insert additional streamlines between these two limits. All these lines must be perpendicular to AB and FG. We now choose an arbitrary number of streamlines within the area arranged so that the seepage passing between any pair is the same as that passing between any other pair. The equipotential lines are also spaced so that the drop in head between any pair is the same as that between any other pair. The resulting "flow net" will then possess the property that the ratio of the sides of each rectangle, bordered by two streamlines and two equipotential lines, is a constant. This means, for example, that some distance m must be approximately equal to the distance n, that other distances such as m_1 must be approximately equal to such distances as n_1, and that $m/n = m_1/n_1$ = a constant. The flow net is usually spoken of as consisting of a system of "curvilinear squares." This is a trial-and-error method in which one must make the streamlines everywhere intersect the equipotential lines at right angles and also produce curvilinear squares. It usually requires more than one attempt to produce a good net. Once the net is established it is possible to compute the quantity of seepage through the medium, the uplift pressure caused by the percolating water, and the pressure gradient at any point.

[6] Forchheimer, Philip, Hydraulik (Teubner, Leipzig), 1930.

The following suggestions are made in order to assist the beginner in employing the graphical method:

1. Study the appearance of all available flow nets regardless of their source.

2. Don't use too many flow channels in your first and second trials. If necessary, additional flow channels can be inserted later.

3. In your first trial observe the appearance of your entire flow net.

4. Use smooth, rounded curves even when going around sharp corners.

5. Make detail adjustments only after the flow net is approximately correct.

Membrane Analogy. The study of flow through granular material in the steady state resolves itself into solving Laplace's differential equation for specific boundary conditions. It can be shown that Laplace's equation also applies to phenomena which are entirely unrelated to fluid flow. The small deflection of a loaded membrane is one of these phenomena, and, by analogy, may be used to solve fluid flow problems experimentally. The Laplace's equation also governs the flow of electricity in homogeneous isotropic media.

Consider a uniformly stretched membrane supported at the edges and subjected to a uniform pressure, P, as shown in figure 16. Then, as in the case of a thin-walled vessel subjected to a uniform internal pressure, the tension in the membrane will be given by the equation,

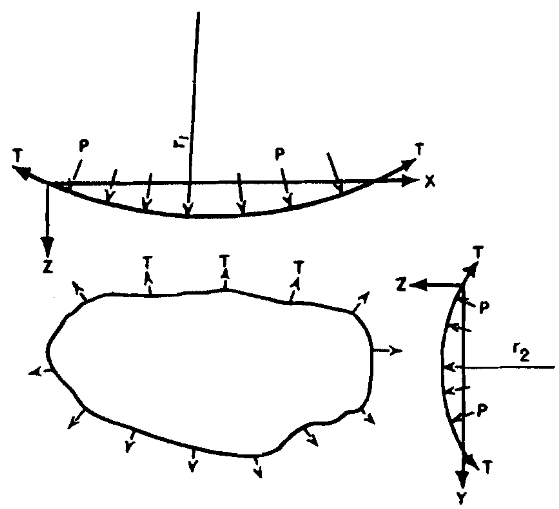


Figure 16 - Uniformly Stretched Membrane Subjected to a Uniform Pressure.

$$\left.\begin{array}{l} \dfrac{T}{r_1} + \dfrac{T}{r_2} = P \\ \text{or} \\ \dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{P}{T} \end{array}\right] \quad \ldots\ldots\ (70)$$

since the tension, T, must be equal in all directions. The curvatures for the membrane, for small deflections, are given by the equations,

$$\frac{1}{r_1} = \frac{\partial^2 z}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ (71)$$

$$\frac{1}{r_2} = \frac{\partial^2 z}{\partial y^2} \quad \ldots\ldots\ldots\ldots\ (72)$$

and substitution of equations (71) and (72) into equation (70) gives

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{P}{T} \quad \ldots\ldots\ (73)$$

Equation (73) must be made to take the form of Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \nabla^2 \phi \quad \ldots\ (74)$$

which is comparable with the equation,

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 = \nabla^2 p \quad \ldots(75)$$

This can be accomplished by performing the experiment without applying the pressure, P; that is, if P is made equal to zero, equations (73) and (74) are of the same form. The membrane then, since it satisfies Laplace's equation, can be used for the determination of streamlines, equipotential lines, or lines of equal pressure in any flow system where the model is subjected to the proper boundary conditions.

Preparation of the Model. The technique described below is that developed by the Bureau. Other procedures could be used

It will be assumed in the following discussion that it is desired to determine the lines of equal pressure in a pervious earth dam resting on a foundation of the same permeability as the dam. A base plate about 1/4 inch in thickness is cut to scale representing the cross-section of the dam and a large portion of the foundation. The amount of foundation to be included should be an area approximately three times as long as the base width of the dam and twice as deep as the reservoir. (See figure 17.)

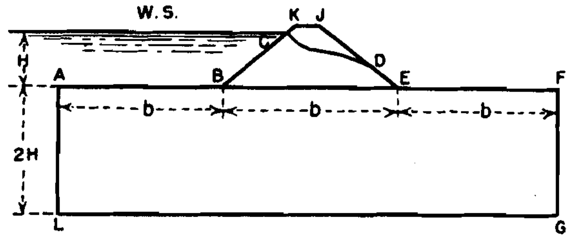


Figure 17 - Membrane Analogy Model.

Around the boundaries of the model is attached a vertical strip of pyralin about 1/16 inch in thickness. Its vertical ordinates are made of a height proportional to the prototype pressure at every point. Referring to figure 17, the strip would have a height of zero along FED. The point D is at first unknown as is the shape of CD, but it is known that the pressure along CD is zero and that it is also a streamline. The exact determination of CD will be discussed in another paragraph. Along CB the vertical strip would vary uniformly from zero height at C, to height H at B. Height H could be made to any convenient scale, say 1/2 inch. If this scale were adopted, the vertical strip along BA would be 1/2 inch higher than along EF or at C. AL would increase in height from 1/2 inch at A to 1-1/2 inches at L, and LG would decrease to 1 inch at G. Finally, the boundary GF would drop in height from 1 inch at G back to zero at F. Note that in this system of boundary conditions, the pressures due to the gravitational potential have been added to the boundary conditions.

Next the model is placed on a table, which is part of a scanning set-up, and leveled. A rubber membrane which has been uniformly stretched and fixed to a frame approximately two feet square, is placed over the model. Steps must be taken to insure that the membrane is everywhere in contact with the boundaries of the model.

It has been found through experimentation that, if the boundary JE is made at zero pressure and no attempt is made to force the membrane into contact with the line KJDE, a line CD automatically develops at zero pressure and the equipotential lines become perpendicular to it. The line

CD is then the desired phreatic line. The model now is fully prepared and its surface can be surveyed. This is done with the device shown in figure 18, which consists of two parallel bars supporting a traveling bar which in turn supports a micrometer depth-gage. The accuracy of the experiment is increased by painting the membrane with a thin coat of varnish and dusting the painted surface with flaked graphite, thus making the membrane an electrical conductor. A 1/8-watt neon glow-lamp connected in series with the micrometer needle and membrane to a 110-volt alternating-current source makes a very sensitive indicator. The exact point of contact between the membrane and the descending micrometer depth-gage point is indicated by the lighting of the neon lamp.

The membrane analogy does not lend itself to experiments in which the percolation factor of the material is different in certain zones than others. It is not quite as accurate nor as rapid an experimental method as the electric analogy. It is also difficult to make the boundary ordinates exactly correct at every point.

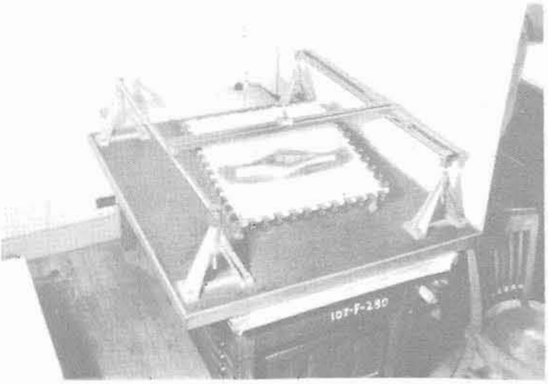


Figure 18 - Membrane Analogy Model.

Electric Analogy. The electric analogy is used to obtain experimental solutions to certain problems arising in the field of hydraulics, particularly in the branch dealing with the slow flow of water through earth masses. It may be applied to both two- and three-dimensional problems. The method consists essentially of producing and studying an analogous conformation, in which the actual flow of water in the soil is replaced with a similar flow of electricity through an electrolyte in a tank or tray that has the same relative dimensions as the earth embankment. This is permissible since Laplace's equation governs both the flow of electricity and the flow of water, where water can be considered a perfect fluid.

The analogy can be seen at once by comparing Ohm's law, which expresses the flow of electricity through a uniformly conductive medium, with Darcy's law, which expresses the flow of water through a homogeneous granular material.

In performing an electric analogy experiment, a model is made of the prototype structure, to scale, so that the prototype boundary conditions are properly represented by boundary conditions in electrical units. It is best to work in terms of potentials, and the method will be better understood if we consider a specific case. Imagine an earth dam with cross-section as shown in figure 19, resting on an impervious foundation. In working with the electric analogy the potential function is usually written in the following form, for it is more convenient to work in units such as feet of head acting or percent of head acting:

$$\phi = p \pm y \quad \ldots\ldots\ldots\ldots\ldots (76)$$

where

p = the pressure-head , and

y = the vertical coordinate of the point.

Darcy's law	Ohm's law
$Q = \frac{KAH}{L}$	$I = \frac{K' A' V}{L'}$
Q = rate of flow of water	I = current (rate of flow of electricity)
K = coefficient of permeability	K' = conductivity coefficient
A = cross-sectional area	A' = cross-sectional area
H = head producing flow	V = voltage producing current
L = length of path of percolation.	L' = length of path of current

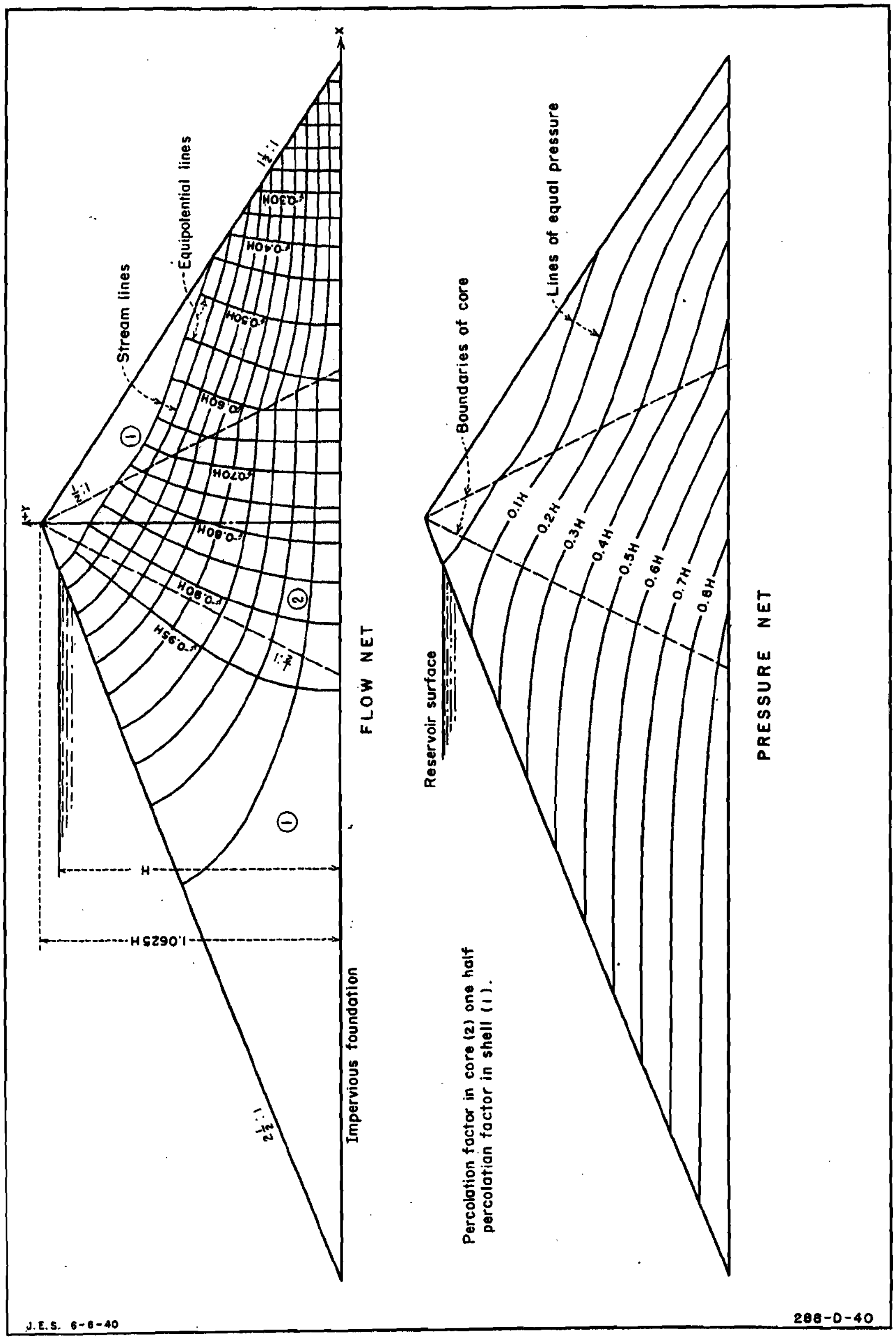


Figure 19 - Electric Analogy Study of Earth Dam.

In this case, also,

$$\left.\begin{aligned} v &= K\frac{d\phi}{ds} \\ Q &= A\ v \end{aligned}\right] \quad \ldots\ldots\ldots\ldots\ldots (77)$$

By use of equation (76) we may establish the boundary conditions for the model. The rectangular coordinate system will be taken, as shown in figure 19, with the origin at the base of the dam and y positive upward. Equation (76) will be used with the + sign. Now, along the upstream face of the dam ϕ will become equal to a constant (H in this case), because everywhere on this face

$$\phi = p + y = H \quad \ldots\ldots\ldots (78)$$

Along the downstream face of the dam where $p = 0$,

$$\phi = + y \quad \ldots\ldots\ldots\ldots\ldots (79)$$

This means that ϕ varies directly with y. The third boundary condition to be met is the establishment of a phreatic line. This line, as mentioned before, is a line of zero pressure and also a streamline. Since p = zero along the phreatic line, from equation (76) we have again

$$\phi = + y \quad \ldots\ldots\ldots\ldots\ldots (79)$$

Mathematically, the boundary conditions are satisfied by equations (76) and (79). These can also be satisfied on the electric analogy model. The determination of the phreatic line is not direct, however, but is a cut-and-try process. It will be discussed in detail hereinafter. The base of the dam, in this example, is a streamline and may be represented by any nonconducting material.

Preparation of the Model. Electric analogy models are usually prepared from pyralin. A thin sheet of pyralin is cemented to a piece of plate glass by the use of acetone. On this plate are erected vertical strips of pyralin along the lines which define, to scale, the cross-section of the dam. These strips are cemented with acetone to the pyralin plate. In the model the constant-potential upstream boundary is represented by a strip of brass or copper which is at a constant electric potential. The base of the dam, which is a streamline, is represented by the pyralin strip, for it is a nonconductor. The downstream face, along which the potential varies, is approximated with a series of small brass or copper strips connected in series with small resistors. The phreatic line, which is also a streamline, is made of modeling clay so that there can be no flow across it, and also so as to facilitate rapid change of its location in the cut-and-try procedure. The original position of the clay boundary representing the phreatic line can be determined by approximation. The experienced operator can estimate its position very closely.

Once the boundaries of the model are prepared the tray is filled with a salt solution or ordinary water, to act as an electrolyte. The electrical circuit is shown on the accompanying drawing, figure 20. The circuit is essentially a Wheatstone bridge, with the model connected in parallel with the main resistor having the variable-center tap. In the cross-circuit the probing needle is connected to the variable-center tap through a small cathode-ray tube which acts as a null-indicator.

Determination of the Phreatic Line. When the model is prepared and set in position, a point is selected on the assumed phreatic line a distance y above the impervious foundation and the potential is read at this point. The potential at this point must equal y, as stated in equation (79), since $p = 0$. Since we are working in percent, we may state that if at the point in question y equals 80 percent of H (where H is the depth of water in the reservoir), then ϕ must equal 80 percent. If ϕ as read on the bridge is not 80 percent, the clay boundary is moved until $y = \phi$. Several points must be checked in this manner until the final determination of the phreatic line is made.

Once the phreatic line is determined, the potentials throughout the model may be read on the Wheatstone bridge. These are plotted as equipotential lines and are shown, for the case discussed, in figure 19. The experimental problem is completed when the equipotential lines are determined. From these lines may be computed the lines of equal pressure, losses of water due to seepage, and pressure gradients. Streamlines, if desired, are drawn perpendicular to the equipotential lines. Lines of equal pressure are computed from the equation

$$p = \phi - y \quad \ldots\ldots\ldots\ldots (80)$$

where values of ϕ are selected from the equipotential net and y is the percent of head, H, at the ϕ point in question. The pressure, p, will also be in percent of head.

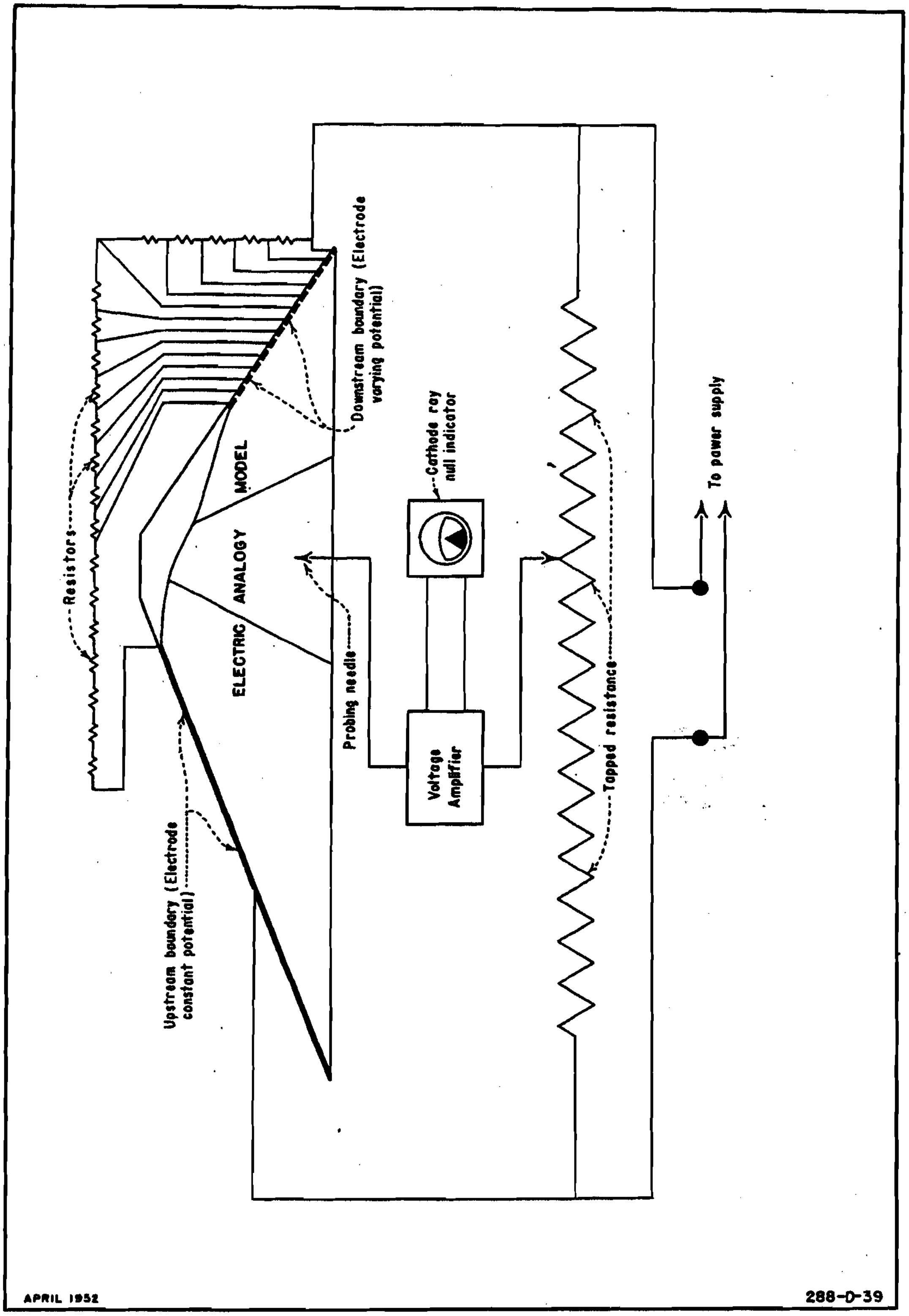


Figure 20 - Electric Analogy Diagrammatic Layout.

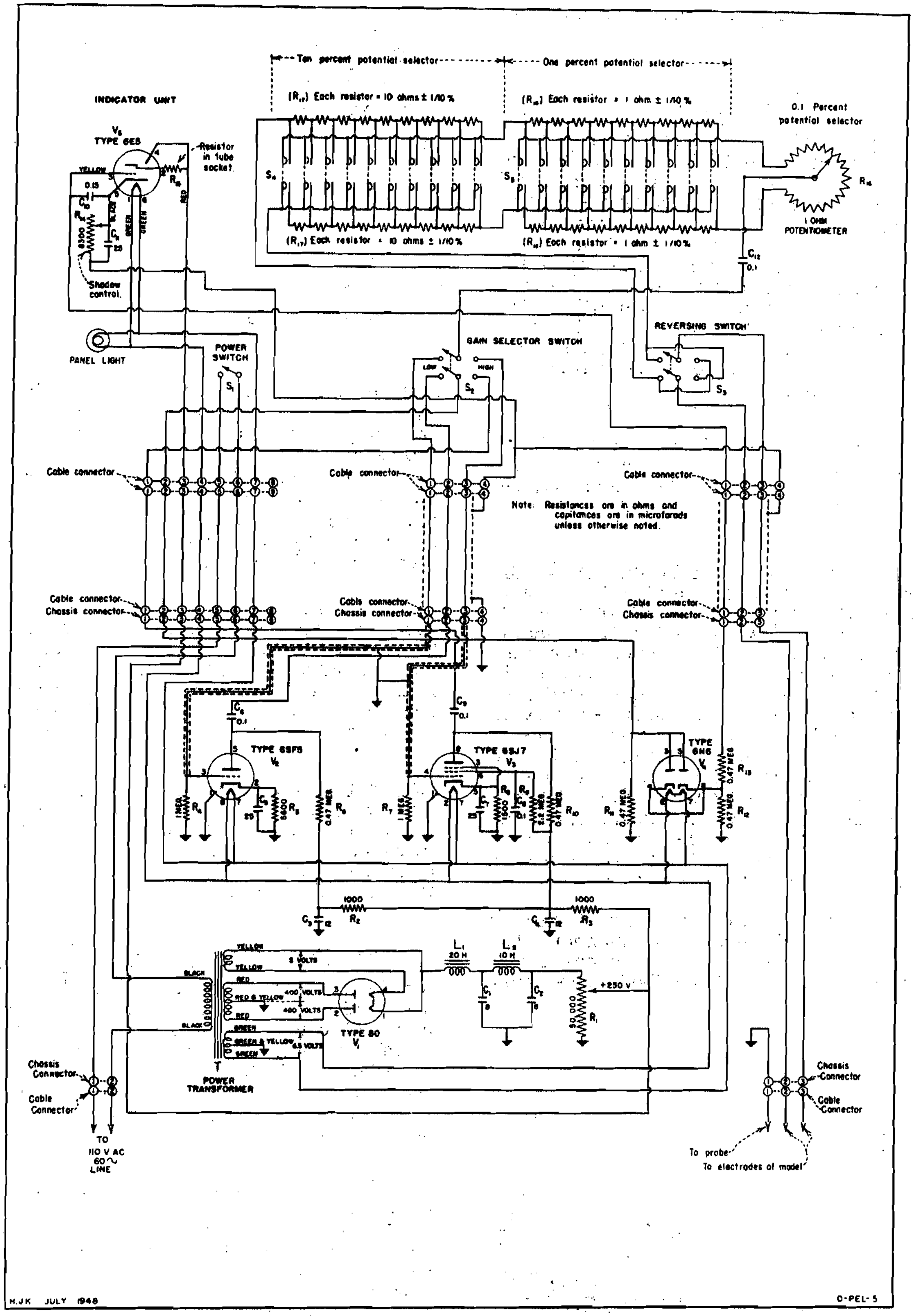


Figure 21 - Circuit Diagram of Selecting, Amplifying, and Indicating Units of the Electric Analogy Apparatus.

The equal-pressure lines are shown in figure 19.

The electric analogy is most readily adapted to problems in which the permeability coefficient (corresponding to the electrical conductivity) is constant throughout the entire soil mass. However, it can be used for problems in which the permeability coefficient is not a constant throughout the entire mass, but is constant in certain regions. In problems of this type, the depths of solutions in the tray are made proportional to the various permeabilities. The experimental results shown on figure 19 are based upon a dam having a core material half as permeable as the material in the outer zones.

For a schematic diagram of the Wheatstone bridge used by the Bureau of Reclamation and a photograph of the equipment, see figures 21 and 22.

An Approximate Solution of a Rapid-Drawdown Problem. In solving a rapid-drawdown problem by the electric analogy tray, the drawdown is considered to be instantaneous and thus the head of water within the dam remains at the full-reservoir water surface elevation. In other words, the point of intersection of the full-reservoir water surface with the upstream face of the dam is the 100-percent potential. The surface from this point along the upstream face to the lowered water surface elevation is considered to be a free surface, and the portion below the lowered water surface is of a potential equal to the lowered elevation divided by the full-reservoir elevation.

The phreatic line is first established for a full reservoir in the usual manner. The upstream, or 100-percent, electrode is then cut down to the elevation of the lowered water

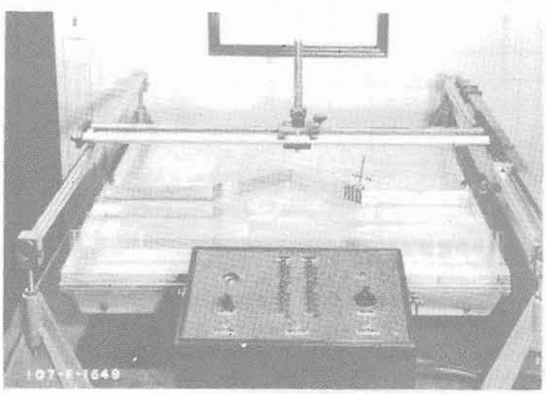


Figure 22 - The Electric Analogy Tray.

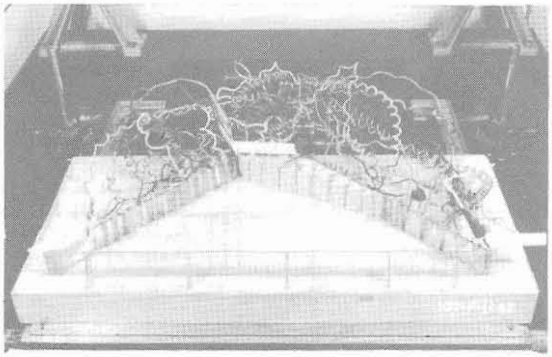


Figure 23 - Model for a Drawdown Problem.

surface and connected to the proper resistance. The remaining resistance is varied uniformly along the upstream face, reaching 100 percent at the entrance of the phreatic line. Wires may be extended from the resistance box used on the downstream face to the resistance strips on the upstream face (see figure 23). The equipotential lines are then surveyed in the usual way.

With the equipotential lines thus established, the streamlines may be drawn, use being made of the fact that the two systems must be orthogonal. Using the equipotential lines, the pressure net may be drawn by the use of equation (80). An example of the flow net and pressure net is shown in figure 24.

Applications of the Electric Analogy. A few practical problems that have been solved by the electric analogy are included here because of their general applicability or because they show the effect of certain conditions on a flow problem.

Green Mountain Dam Study. This was an electric analogy study made for the determination of uplift pressures and the flow net existing in the dam. It is included here because it demonstrates that in a zoned dam in which one zone is of relatively impervious material, practically all the head losses will occur in this material even though the water has previously passed through a relatively pervious zone.

Figure 25 shows the flow net and the pressure net that will exist in the dam for the section studied. Figure 26 shows two photographs of the models used in the experiments. Salt solutions of different concentrations were used in the experiment to represent different permeability coefficients. This procedure was abandoned later in favor of the method of varying the

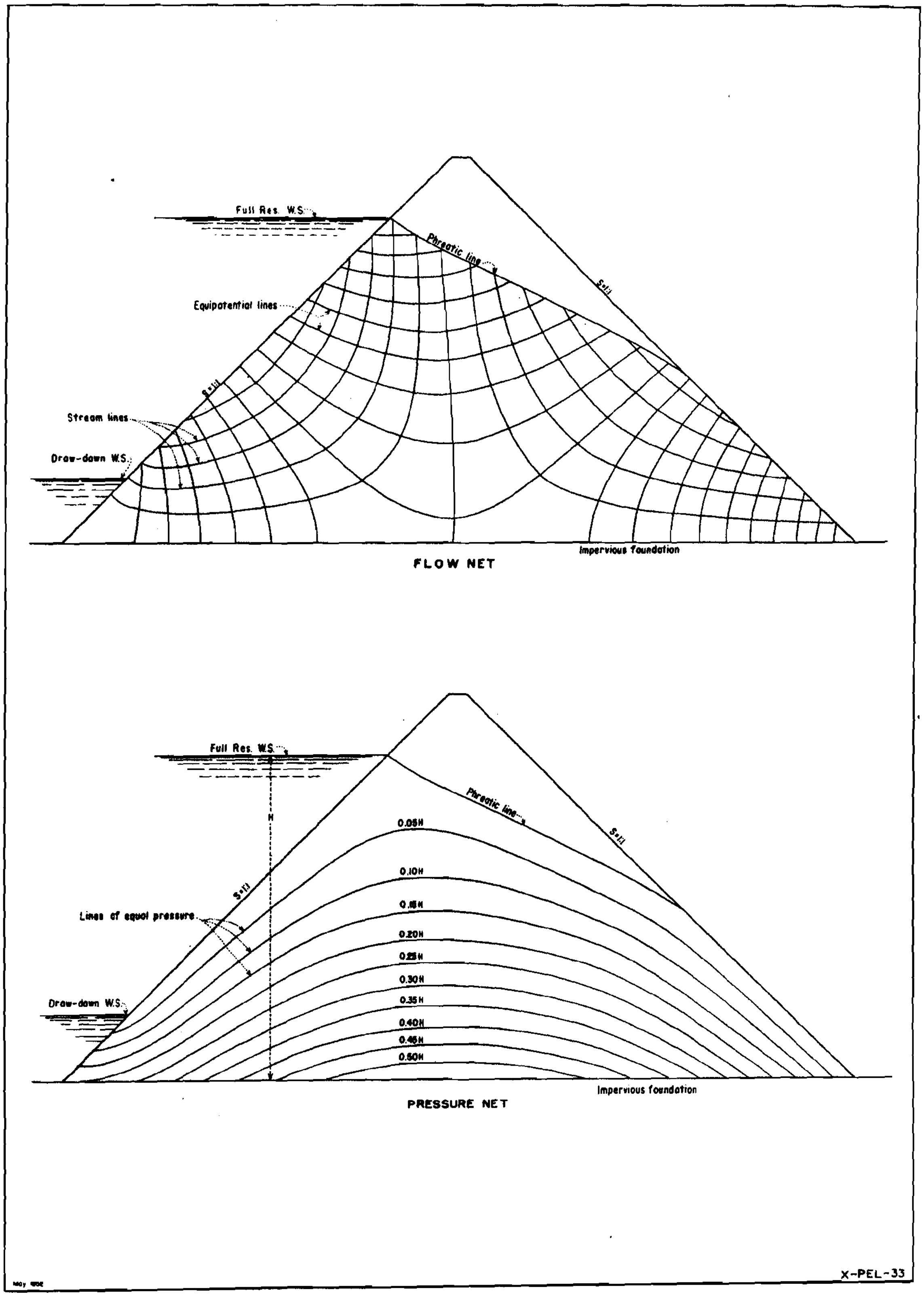


Figure 24 - Rapid-drawdown Flow and Pressure Nets for 1:1 Upstream and 1:1 Downstream Slopes, Homogeneous and Isotropic Material.

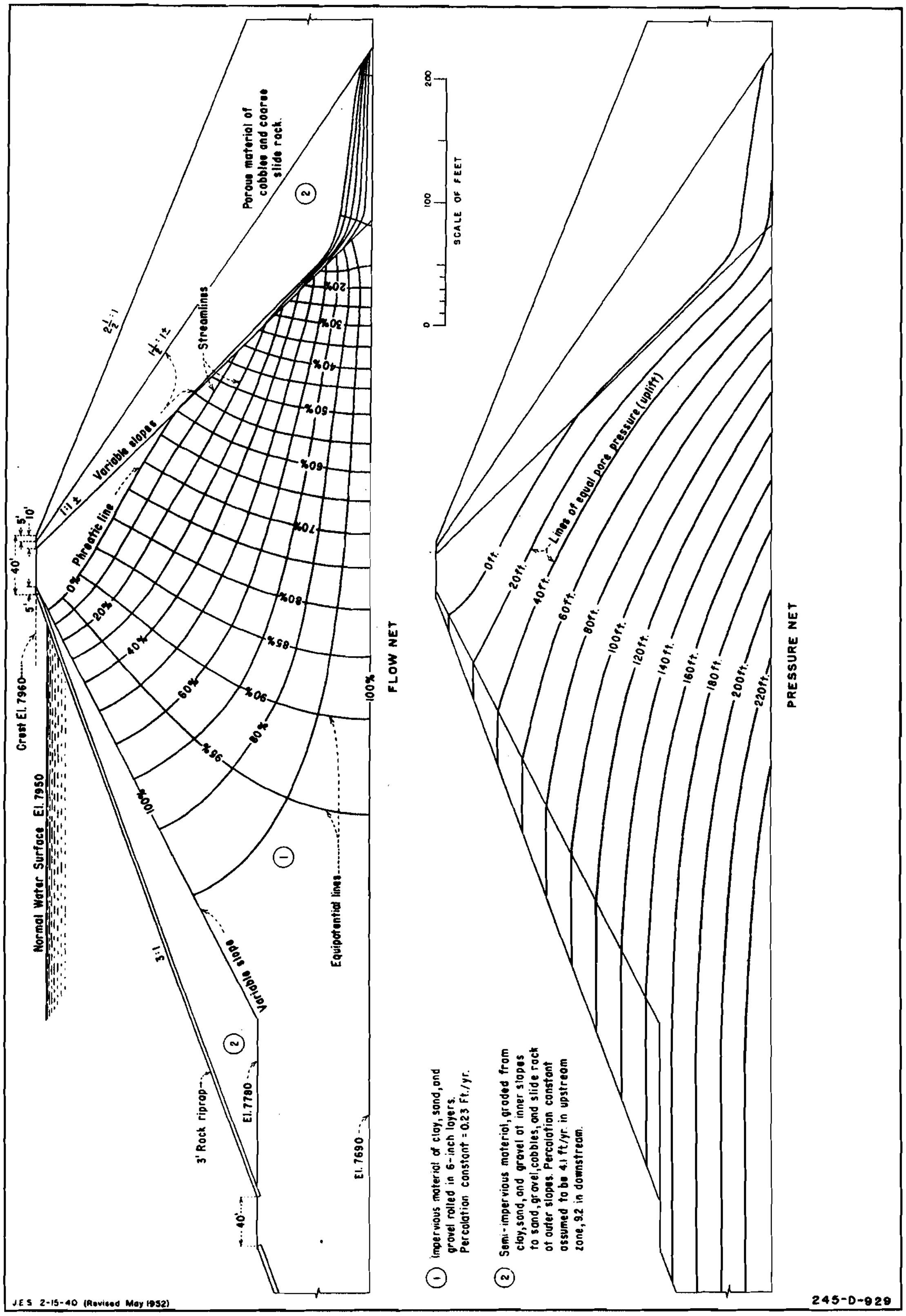


Figure 25 - Electric Analogy Study of Green Mountain Dam, Colorado-Big Thompson Project.

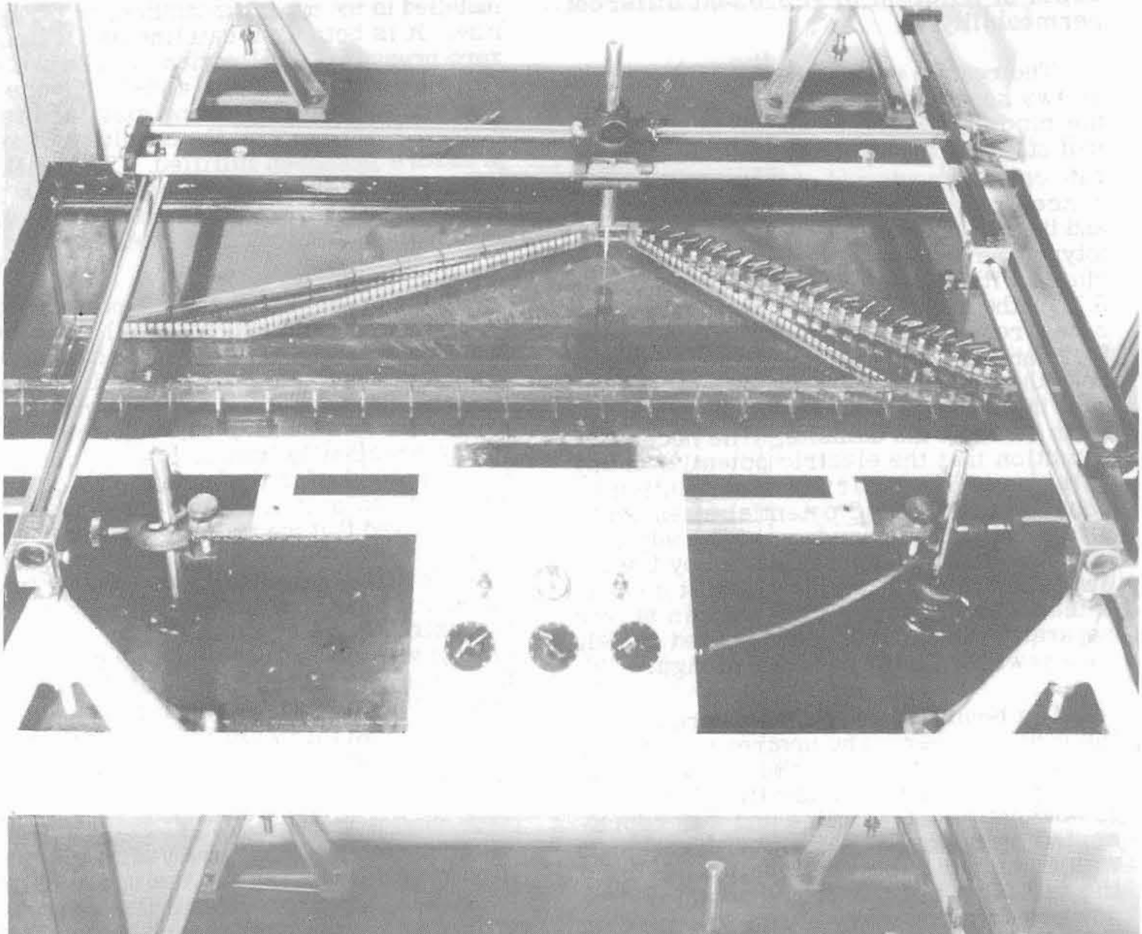


Figure 26 - Green Mountain Electric Analogy Models. (Top: Original Model. Bottom: After Modification.)

depth of solution to represent different permeability.

The results presented herein are based on two separate approaches. In one case, the model of which is shown in the upper half of figure 26, electrolytes in three separate compartments and of three distinct salt concentrations represent the inner zone and the two adjacent outer zones. In the prototype, the permeability coefficients associated with these zones are 0.23, 4.1, and 9.2 for the inner, upstream, and downstream zones, respectively. These are in terms of feet per year per unit gradient. For each zone these permeability coefficients represent the average values obtained from soil tests made at the damsite. The necessary condition that the electric potential at any point on the boundary of one solution be equal to the electric potential at an opposite point on the boundary of the adjacent solution has been approximated by the installation of small strips of sheet copper. These were bent over the pyralin sheets separating the solutions, and spaced closely as shown in the upper half of figure 26.

The boundary conditions were met in the usual manner. The upstream boundary is a line at constant potential and was made of copper. The rock foundation base-line is obviously a streamline and was represented by means of a strip of pyralin. It was assumed to be at Elevation 7690 for the full length of the cross-section. The downstream boundary of the downstream Zone 2 (see figure 25) is a line of uniformly varying potential, providing that the permeability coefficient of the next downstream zone is quite large by comparison. This condition is fulfilled in this case. The varying electric potential along the boundary was obtained by placing along it 26 equally spaced pieces or segments of copper, which were connected with a series arrangement of 25 one-ohm resistors, as shown in the photograph. When the top segment, which has its centerline at reservoir level, is connected to the upstream copper boundary, and the Wheatstone bridge is connected across the upstream copper boundary and the downstream segment at foundation level, the necessary electrical connections are in order. This method of approximating the varying electric potential boundary by finite increments does not give satisfactory results unless the current going through the resistors is large relative to the current going through the solutions. This will be the case when the over-all resistance of the solutions is large relative to the total resistance of the varying potential boundary.

The remaining boundary condition to be installed is the upper streamline, or phreatic line. It is both a streamline and a line of zero pressure. Its location is unknown and must be found by a cut-and-try process. It is formed with modeling clay, and is in correct location when the condition of zero pressure has been fulfilled. This will be when the potential as measured with the Wheatstone bridge varies linearly with elevation changes along this line.

In the process of fixing the phreatic line and surveying the equipotential lines, two facts became apparent. It was obvious that there was no detectable voltage drop in the upstream salt solution and only a negligible drop in the downstream salt solution. Also, it was practically impossible to adjust the clay correctly for the phreatic line in the downstream salt solution. Results of the tests showed that the problem could be better and more adequately handled by using only a single salt solution for the central or inner zone, and moving the copper boundaries to the extremities of this inner zone. This change was made on the model as shown in the lower photograph of figure 26. The upstream boundary of the inner zone was then held at constant potential, and the downstream boundary at uniformly varying potential.

Equipotential lines surveyed on the modified model are shown in the upper half of figure 25. The streamlines have been drawn orthogonal to the equipotential lines. The pressure net has been obtained from the equipotential system by subtracting from it the elevation component. The nets have been continued through the downstream zone. The probable position of the free surface in this zone has been obtained mathematically by assuming that in the major portion of this zone the phreatic line is a straight line, and by equating the quantity of water passing this zone to the quantity computed from the flow net in the inner zone.

<u>Debenger Gap Dam Study</u>. This electric analogy study is included because it demonstrates the effect of several materials of different permeability on the flow net and pore-pressure distribution in a dam and its foundation. Two cross-sections of the dam and foundation were studied as shown in figures 27 and 28. Note that the dam has a tight, impervious material in its center zone (K = 1.0 foot per year) flanked upstream and downstream by a relatively pervious material (K = 10.0 feet per year) with additional rock-fill material on the downstream face of the dam. The downstream rock fill is an excellent filter which

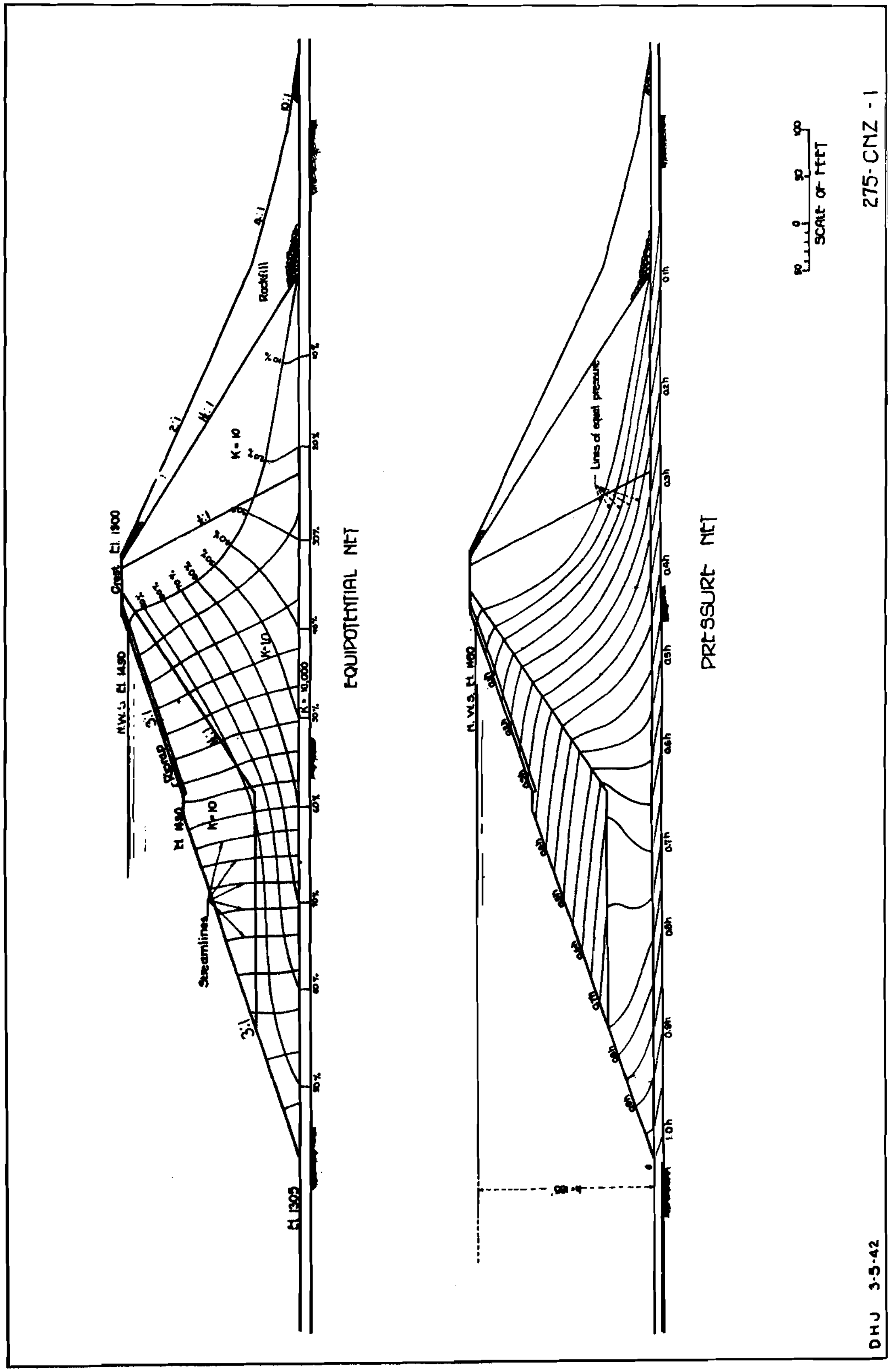


**Figure 27 - Electric Analogy Study of Debenger Gap Dam.
Section in River Channel.**

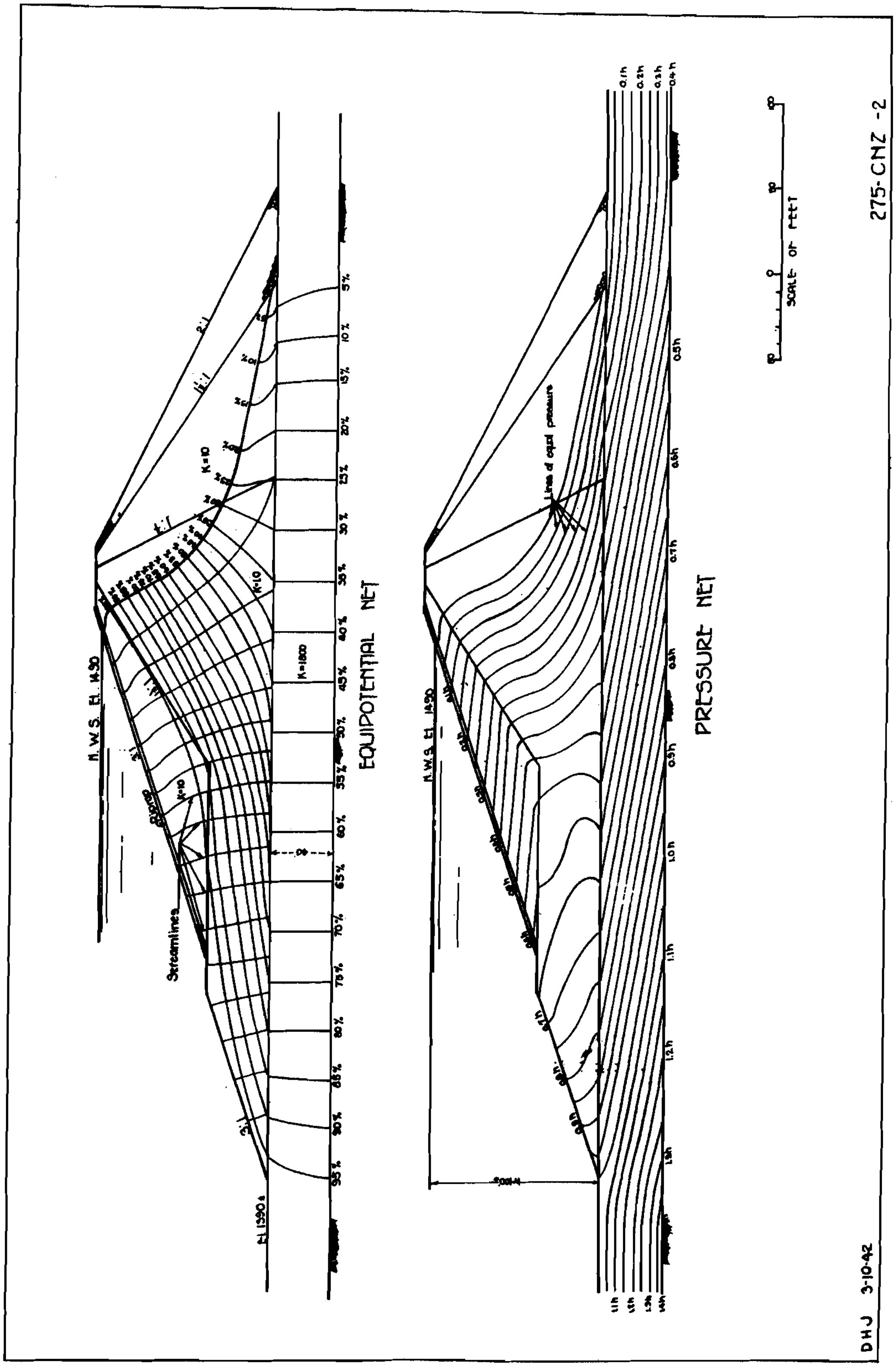


Figure 28 - Electric Analogy Study of Debenger Gap Dam. Section on Left Abutment.

relieves the pore pressure along its boundaries and prevents high exit gradients. The zoning of the materials, in general, is considered very good. By having a pervious material upstream as well as downstream in the prototype, the internal pore pressures would be almost immediately relieved for a rapid drawdown of reservoir.

The experiment had to be performed in two distinct steps due to the great difference in the permeability of the foundation material and the materials within the dam. First, the foundation was treated separately and the equipotential net established. The potentials thus established were then imposed upon the base of the dam for use in determining the equipotentials for the dam itself. The differences in permeability of the materials in the dam were provided in the model by having a depth of solution 10 times as great in the outer zones as in the inner zone.

The results of the study are shown in figures 27 and 28.

Davis Dam Study. The purpose of this electric analogy study was to determine the pore pressures due to the percolating water, and the effectiveness of sheet-pile cut-offs and a clay upstream-toe blanket in reducing the water losses from seepage. The cross-section of the dam, with permeability coefficients for the various materials, is shown in figure 29. Note that the general scheme of zoning materials is much like that used for Debenger Gap Dam.

The first step in the procedure was to study the dam and foundation shown in figure 30, which has no cut-off wall or clay blanket. Water losses and pore pressures were then computed for this condition and compared with results obtained for other assumed conditions.

Conditions assumed and studies made were as follows:

1. No cut-off wall--no clay blanket. (See figure 30.)

2. Cut-off wall extending to bedrock, with 1/32-inch openings between 16-inch sheet-piles.

3. Cut-off wall extending nine-tenths of the depth to bedrock. (See figure 31.)

4. Cut-off wall extending eight-tenths of the depth to bedrock. (See figure 32.)

5. Cut-off wall extending seven-tenths of the depth to bedrock. (See figure 33.)

6. Cut-off wall extending five-tenths of the depth to bedrock. (See figure 34.)

7. Clay blanket extending 315 feet upstream from core of dam. (See figure 35.)

8. Clay blanket extending 515 feet upstream from core of dam. (See figure 36.)

Table 2 consolidates the information obtained. Note that a cut-off wall of depth equal to nine-tenths the thickness of the foundation material reduces the percolation losses by only 23 percent. Also note that if sheet-pilings have joint openings of as little as 1/32 of an inch, they are almost totally ineffective in reducing percolation losses.

A 315-foot clay blanket on the upstream toe reduces the percolation through the foundation material by an amount equal to the reduction caused by an impermeable cut-off wall of depth equal to nine-tenths the depth of the permeable foundation. In addition, the clay blanket without cut-off walls gives the most favorable distribution of uplift pressures for stability calculations.

The amount of percolation through the clay core is shown on figure 37. In comparison with percolation through the foundation material, the percolation through the core is extremely insignificant, and the width of the core may therefore be decreased if desired.

Figure 37 indicates a rapid increase of the percolation gradient near the downstream intersection of the core with the toe blanket. This increase in the percolation gradient could be effectively reduced by a clay fillet between the core and the downstream-toe blanket.

If the 60-foot deep clay cut-off section in the excavation portion of the foundation (ABCDA in figure 29) were replaced by a clay lens with an average depth of 5 feet and the same total length of 645 feet, located at or near the original streambed, the total underflow would be increased to only 8.6 second-feet (or 43 percent over Condition 1) with a considerable decrease in excavation and fill requirements.

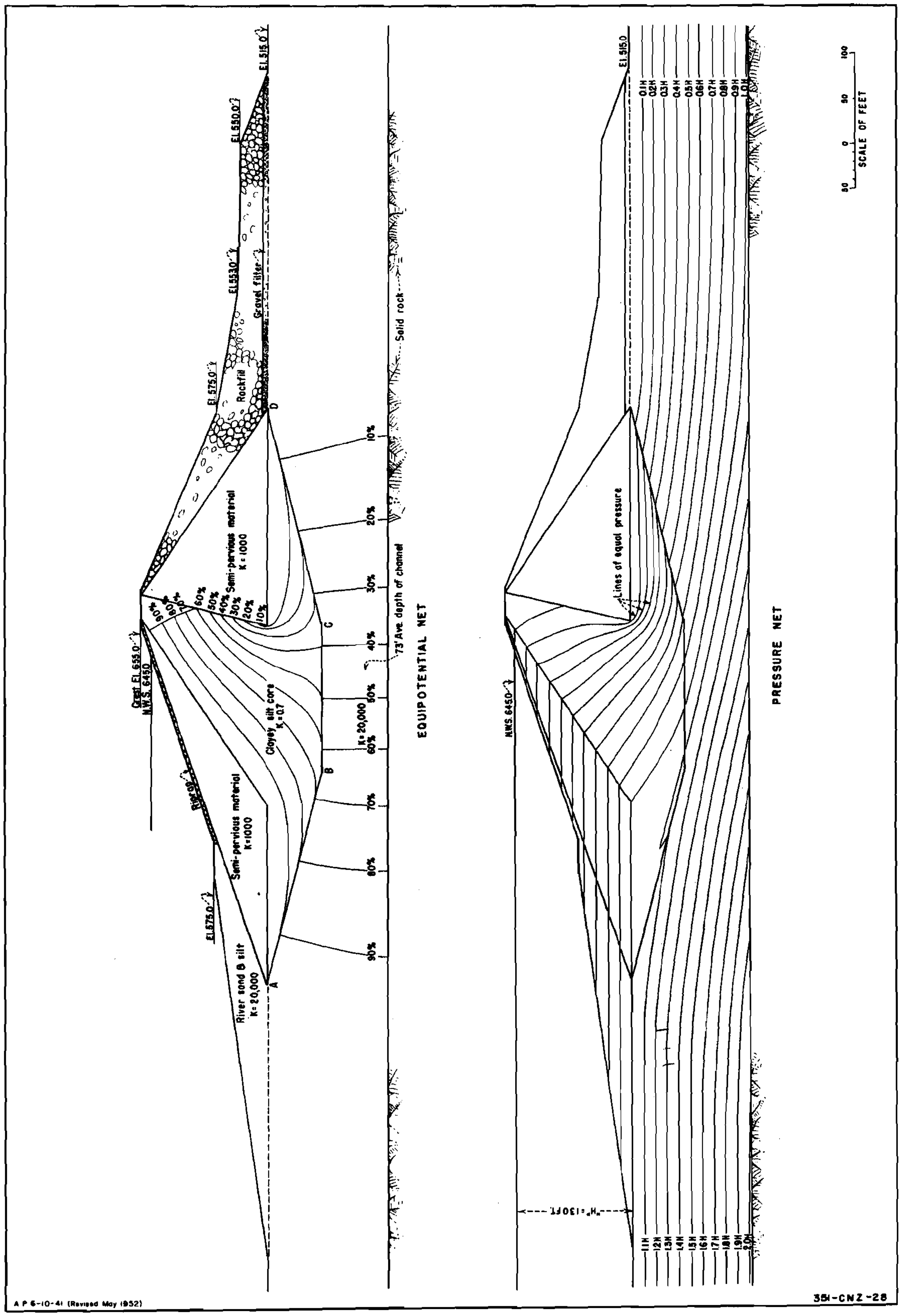


Figure 29 - Electric Analogy Study of Davis Dam without Cut-off Wall or Clay Blanket. Nets for Dam and Foundation Superimposed.

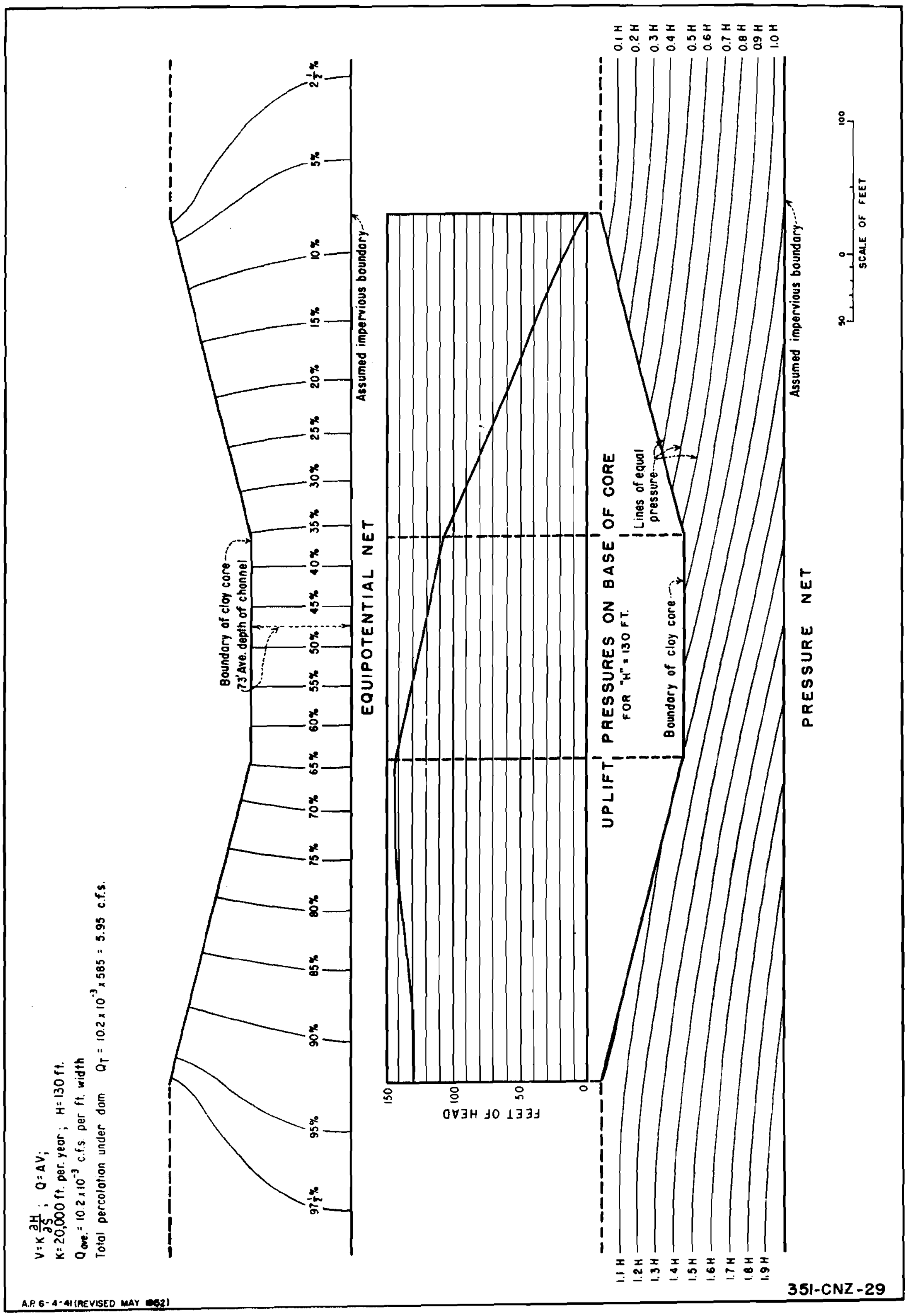


Figure 30 - Electric Analogy Study of Davis Dam without Cut-off Wall or Clay Blanket.

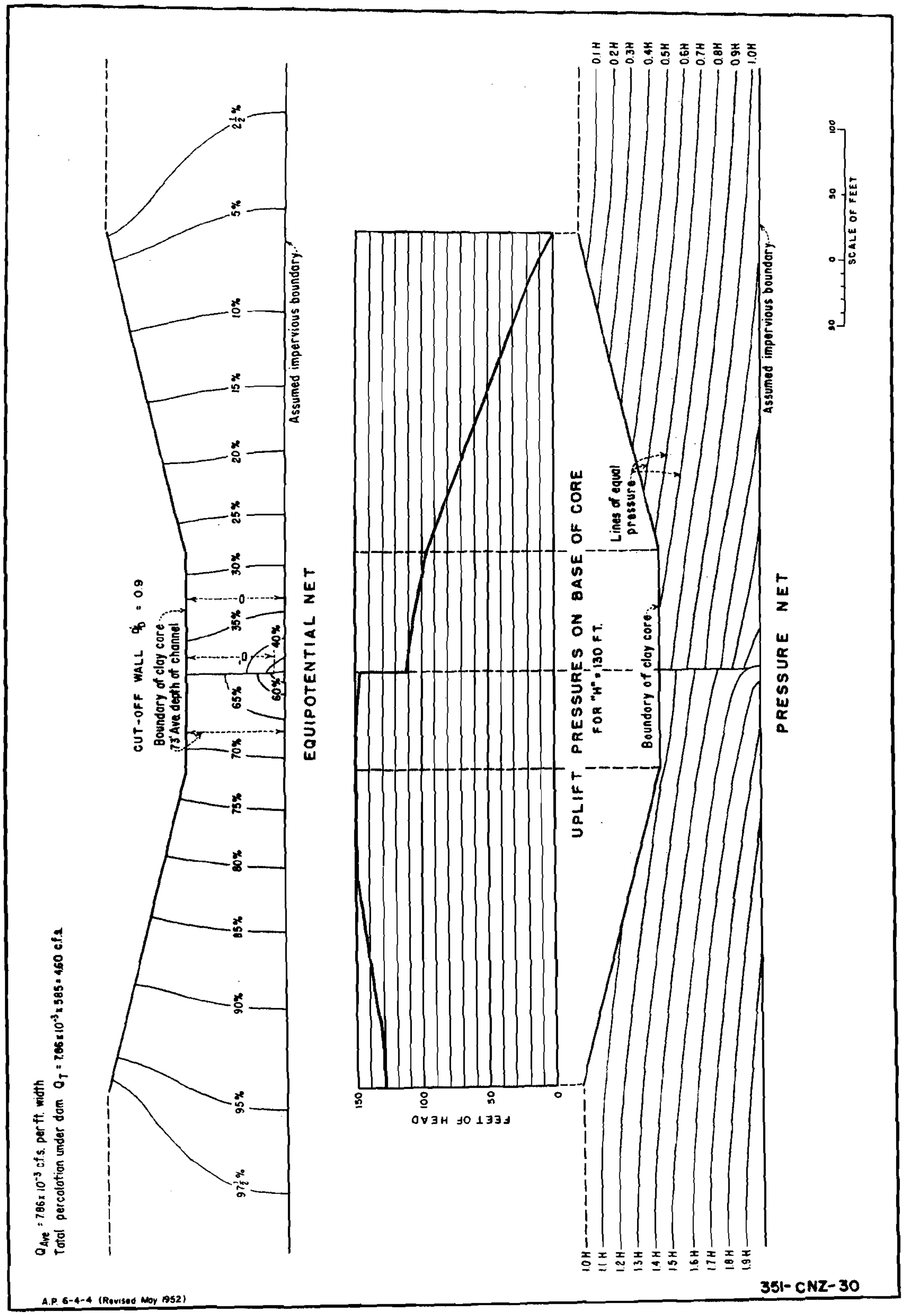


Figure 31 - Electric Analogy Study of Davis Dam. Cut-off Wall for D'/D = 0.9.

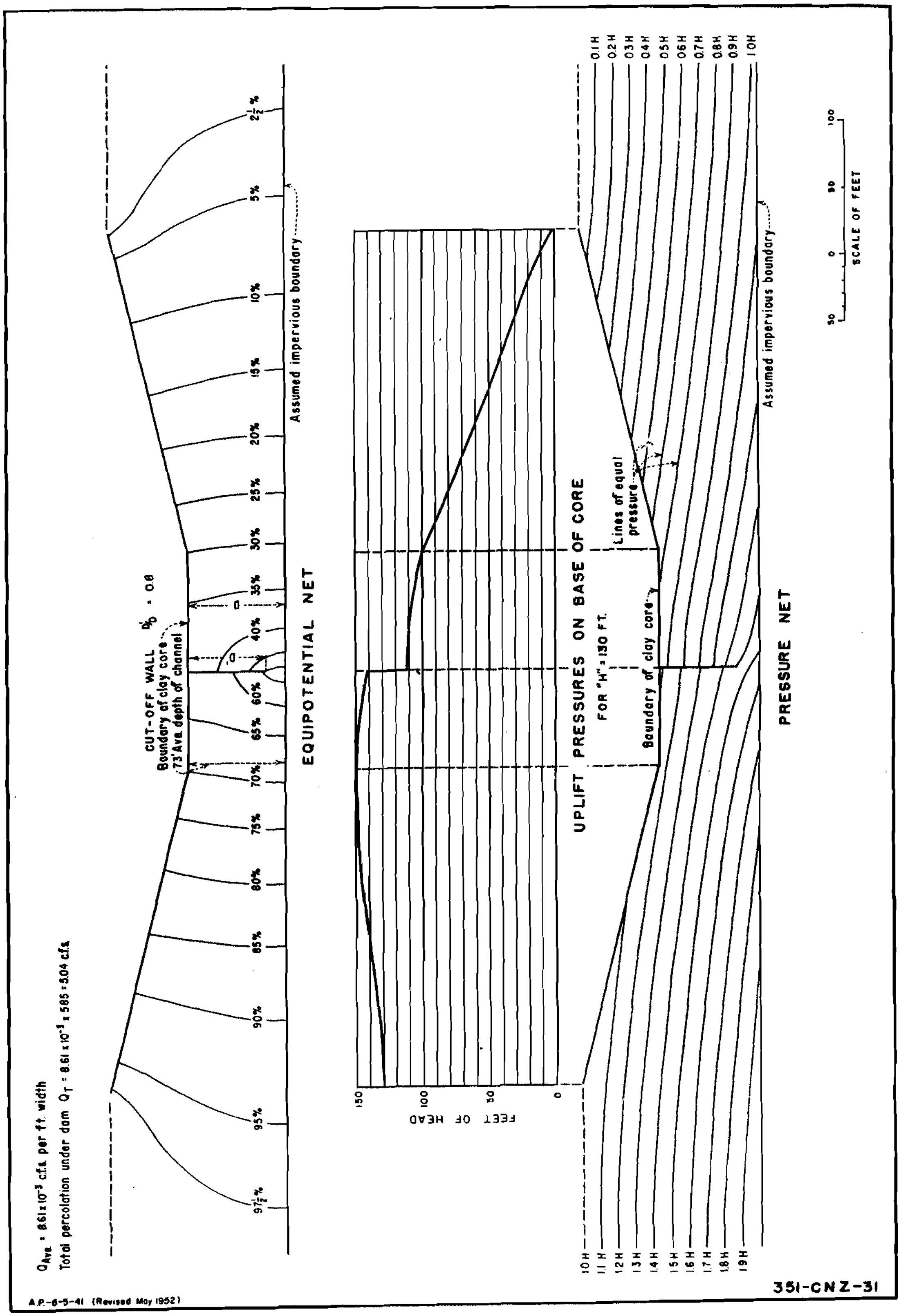


Figure 32 - Electric Analogy Study of Davis Dam. Cut-off Wall for D'/D = 0.8.

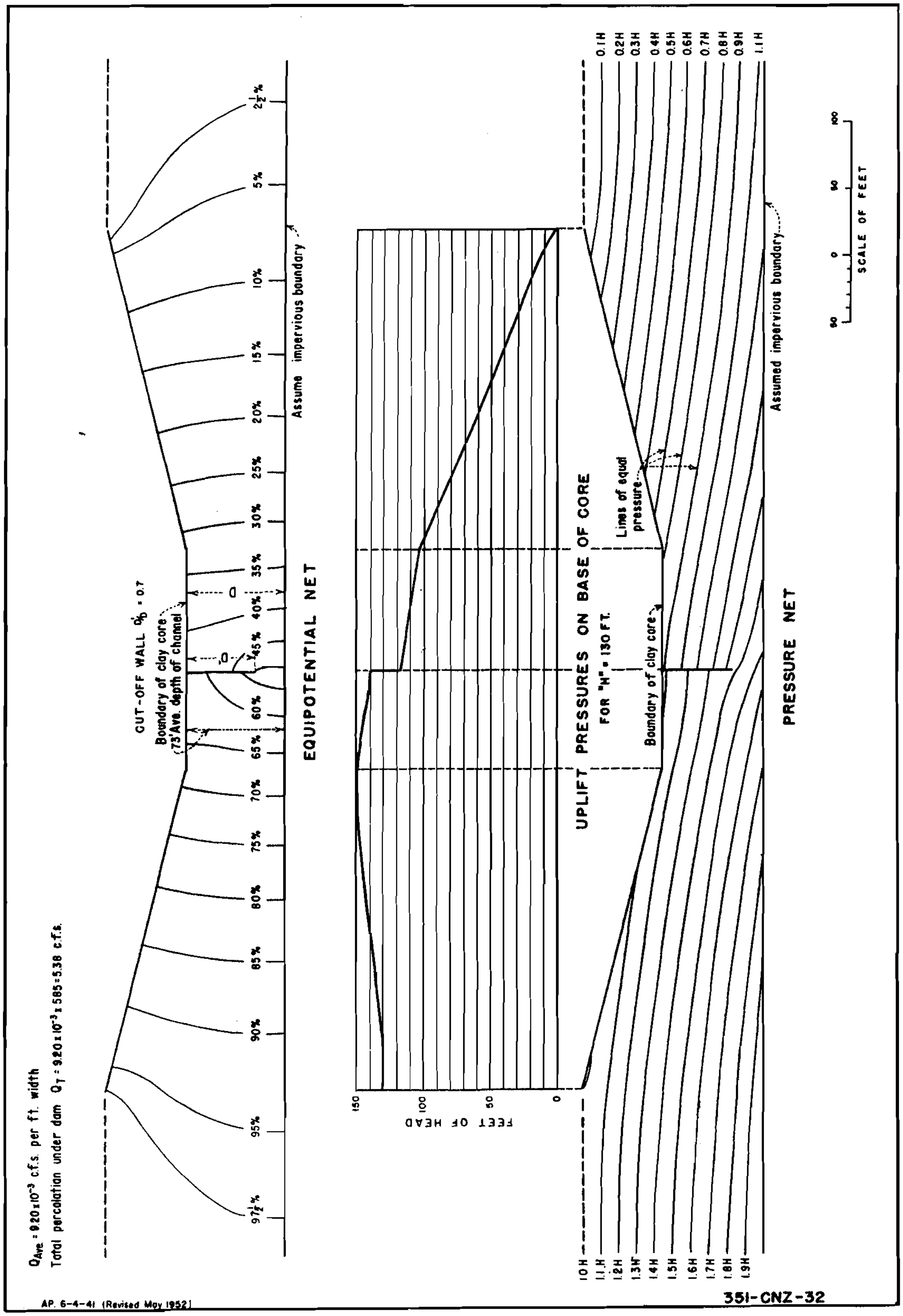


Figure 33 - Electric Analogy Study of Davis Dam. Cut-off Wall for D'/D = 0.7.

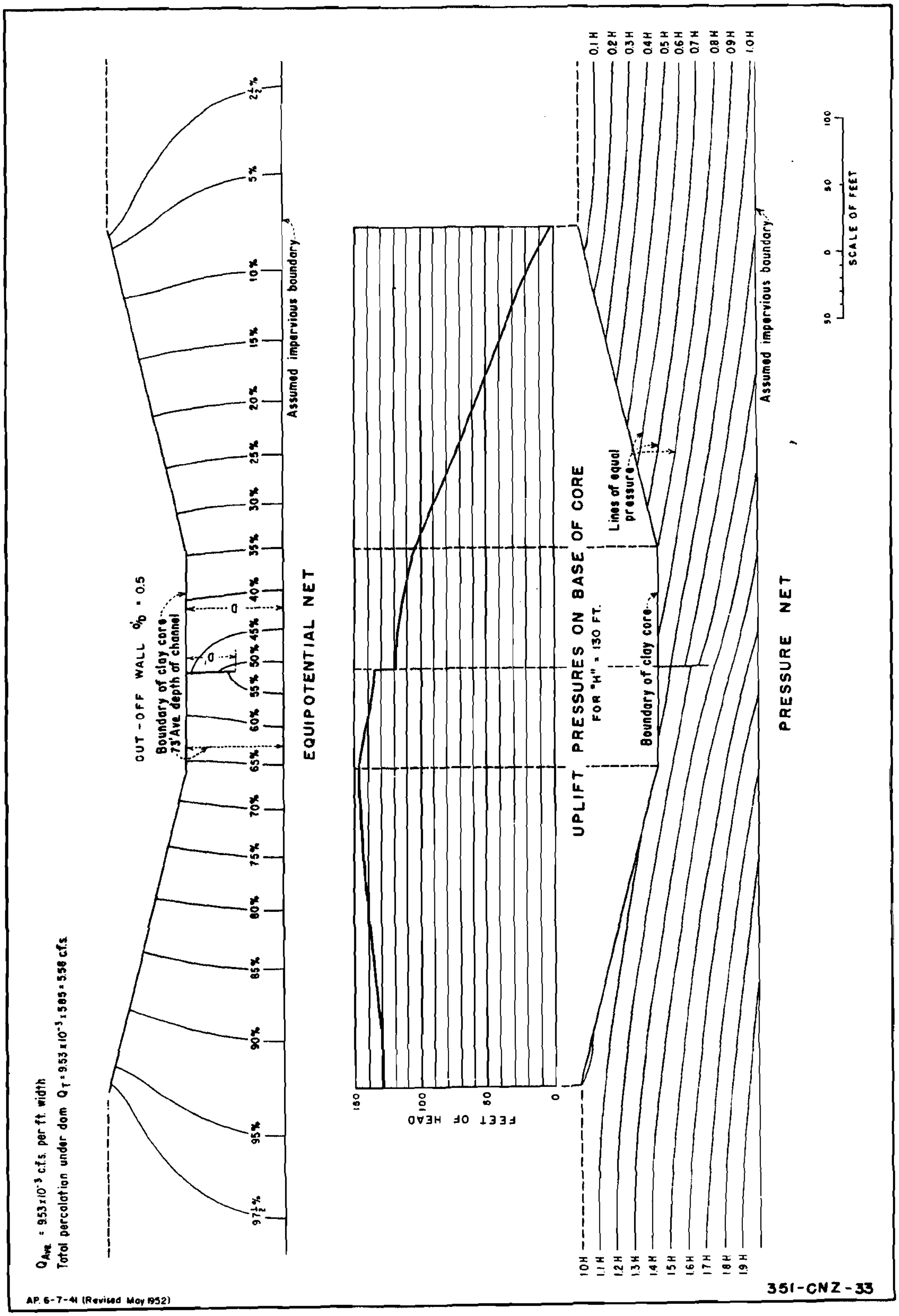


Figure 34 - Electric Analogy Study of Davis Dam. Cut-off Wall for D'/D = 0.5.

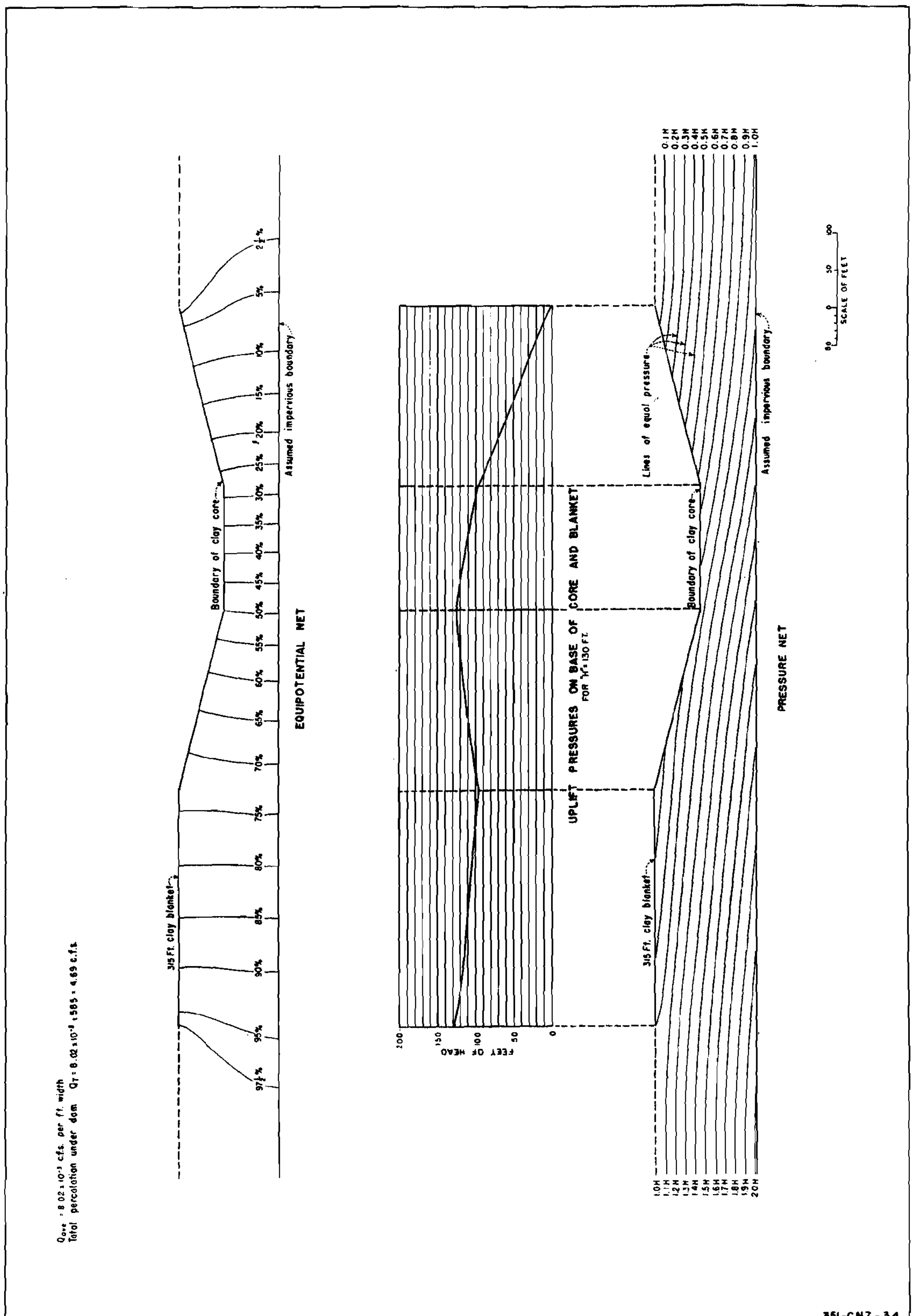


Figure 35 - Electric Analogy Study of Davis Dam. 315-Foot Clay Blanket on Upstream Toe.

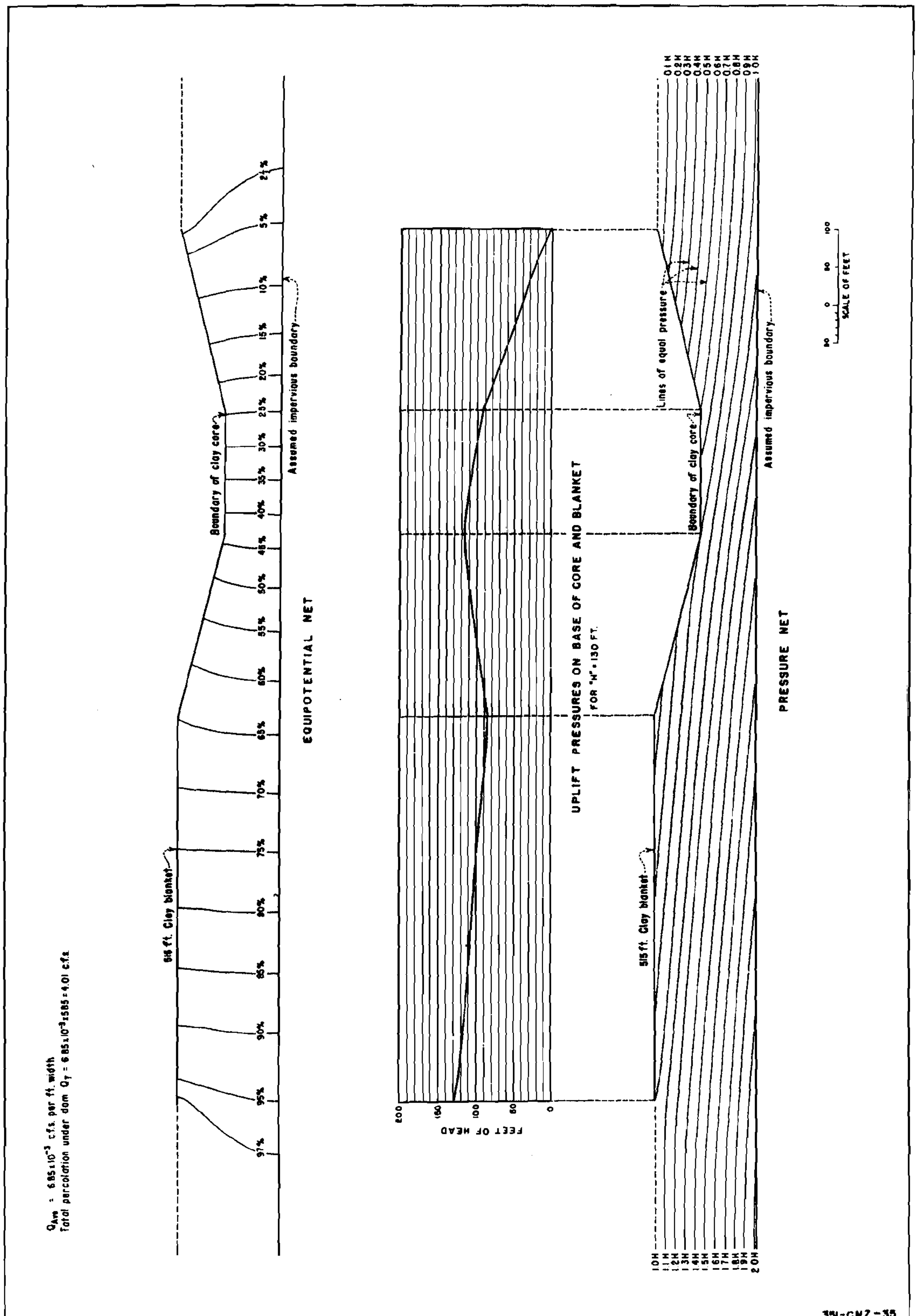


Figure 36 - Electric Analogy Study of Davis Dam. 515-Foot Clay Blanket on Upstream Toe.

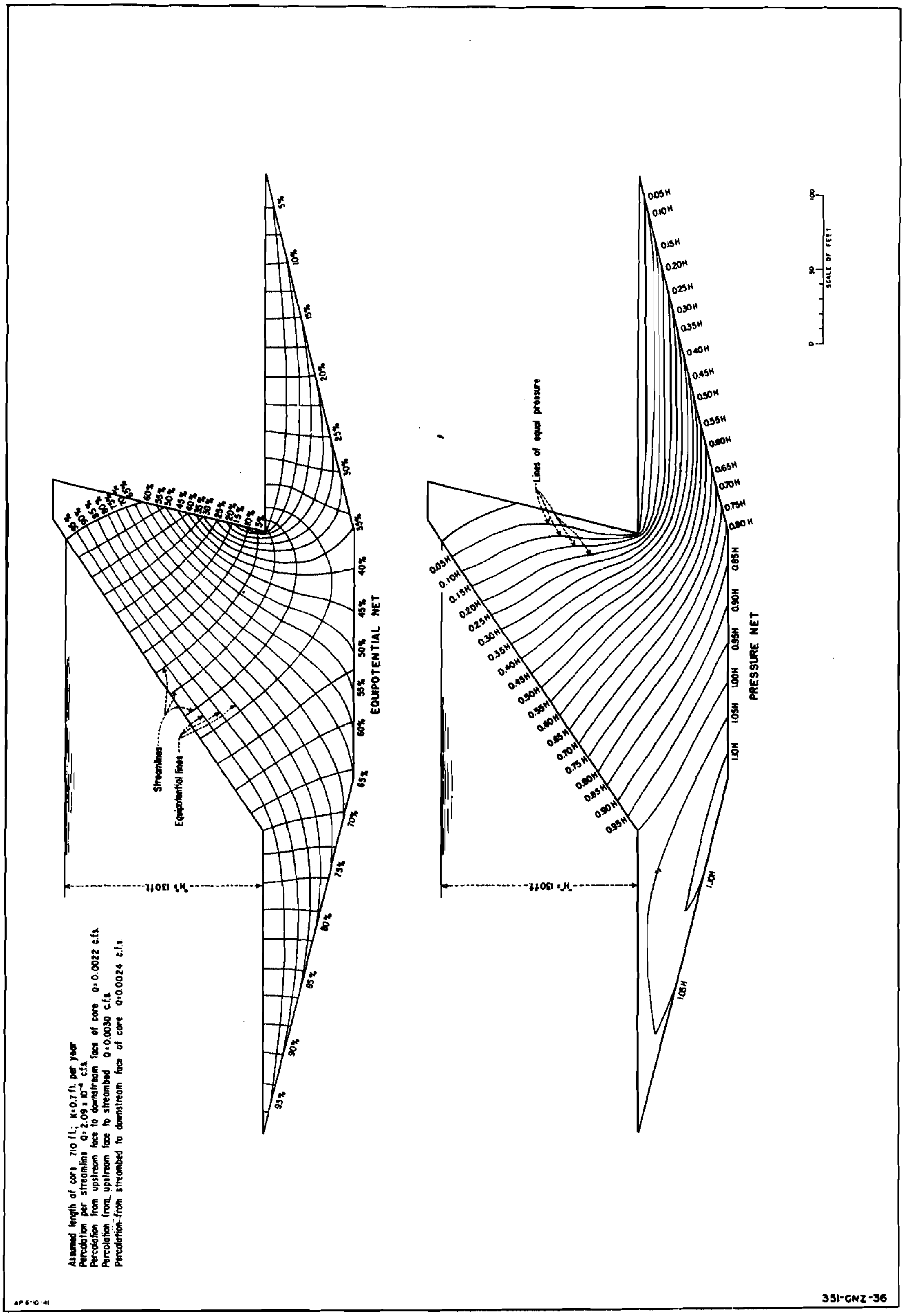


Figure 37 - Electric Analogy Study of Davis Dam. Clay Core.

TABLE 2

RESULTS OF DAVIS DAM PERCOLATION STUDIES

Conditions	Average underflow per foot width of streambed, second-feet	Total underflow, second-feet	Underflow in percent of underflow for Condition 1
1. No obstruction (fig. 30)	10.20×10^{-3}	5.95	100
2. Sheet-piling, 1/32-inch opening between 16-inch piles	10.10×10^{-3}	5.90	99
3. Cut-off wall, D'/D = 0.9 (fig. 31)	7.86×10^{-3}	4.60	77
4. Cut-off wall, D'/D = 0.8 (fig. 32)	8.61×10^{-3}	5.04	85
5. Cut-off wall, D'/D = 0.7 (fig. 33)	9.20×10^{-3}	5.38	90
6. Cut-off wall, D'/D = 0.5 (fig. 34)	9.53×10^{-3}	5.58	94
7. 315-foot clay blanket on upstream toe (fig. 35)	8.02×10^{-3}	4.69	79
8. 515-foot clay blanket on upstream toe (fig. 36)	6.85×10^{-3}	4.01	67

Hydraulic Models. Hydraulic models may be used to determine the flow properties of a hydraulic structure. It is possible to determine streamlines and equal-pressure lines directly, and equipotential lines can be located from either pattern. There are two distinct types of hydraulic experiments employed in the study of slow flow through granular materials. One experimental procedure is known as the "Viscous-Fluid Method." The other may be called the "Hydraulic Scale-Model Method." Only the former is described herein.

Viscous-Fluid Method. The Viscous-Fluid Method is one that gives very rapid results with a minimum of equipment. It is best suited for the determination of streamlines under a weir or diversion dam resting on a pervious material, in which case the structure itself is considered impervious. The effect of cut-off walls extending below the structure is clearly shown.

To perform the experiment, a small tank is constructed with parallel sides of glass plate spaced a small distance apart. A model of the cross-section of the dam with its protruding cut-off walls is then cut from a material such as bakelite, the thickness of the model being such that the model fits snugly between the glass plates of the tank. Before inserting the model between the plates, it is greased to insure watertightness. Water is then made to flow slowly under the model from the upstream to downstream end of the structure, and after the flow has become steady a permanganate solution is added. It will flow in distinct streamlines within the water, as shown for a weir in (a) of figure 38. Pressure lines can be determined from the streamlines.

Experimenters have also used glycerine as the fluid in place of water in performing this type of experiment. In this procedure dyed glycerine is introduced at specific points. This colored glycerine will flow along with the plain glycerine in well-defined streamlines. A photograph of the apparatus under test is shown in (b) of figure 38.

PERMEABILITY OF MATERIALS

General. In order to determine the seepage losses through concrete and earth dams, or

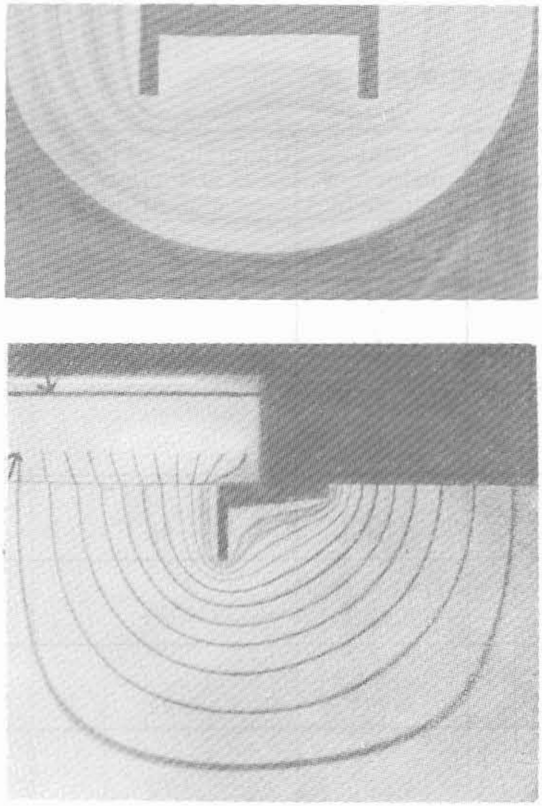


Figure 38 - Streamlines Obtained by Viscous-Fluid Method. (Top: Streamlines under a Weir. Bottom: Streamlines under a Cut-off Wall.

through other earth masses, one must know or be able to determine the permeability coefficient of the soil or concrete. The permeability coefficient of concrete and soils covers a wide range. Values of K obtained from various sources are given in Tables 3 and 4.

Concrete Permeability. The Bureau of Reclamation has made some extensive tests on the permeability of concrete [7]. A summary of the results obtained from these tests is given here with an example showing application of the data.

Figure 39 shows how the permeability of a concrete specimen varies with the length of test specimen. Figure 40 gives the variation of the permeability with water-cement ratio and maximum-size aggregate. The figures adjacent to the small circles indicate the number of cylinders in each series of tests.

The conclusions of the tests are enumerated in the eight paragraphs below:

1. Percolation of water through mass concrete follows the normal laws of viscous flow as expressed by an equation of the form $\frac{Q}{A} = K \frac{H}{L}$ (Darcy's law) in which K is the permeability coefficient or the unit rate of discharge at unit hydraulic gradient. For use in the above the coefficient as obtained in tests on laboratory specimens must be properly corrected for specimen end-effect.

2. For the concrete tested, the major factors controlling permeability were water-cement ratio and maximum size of aggregate. For water-cement ratio values ranging from 0.45 to 0.80 by weight, the corresponding range in permeability for mixes containing the same maximum size of aggregate was

[7] Ruettgers, A., Vidal, E., and Wing, S. P., "Permeability of Mass Concrete," Proceedings, ACI, Vol. 31, 1935.

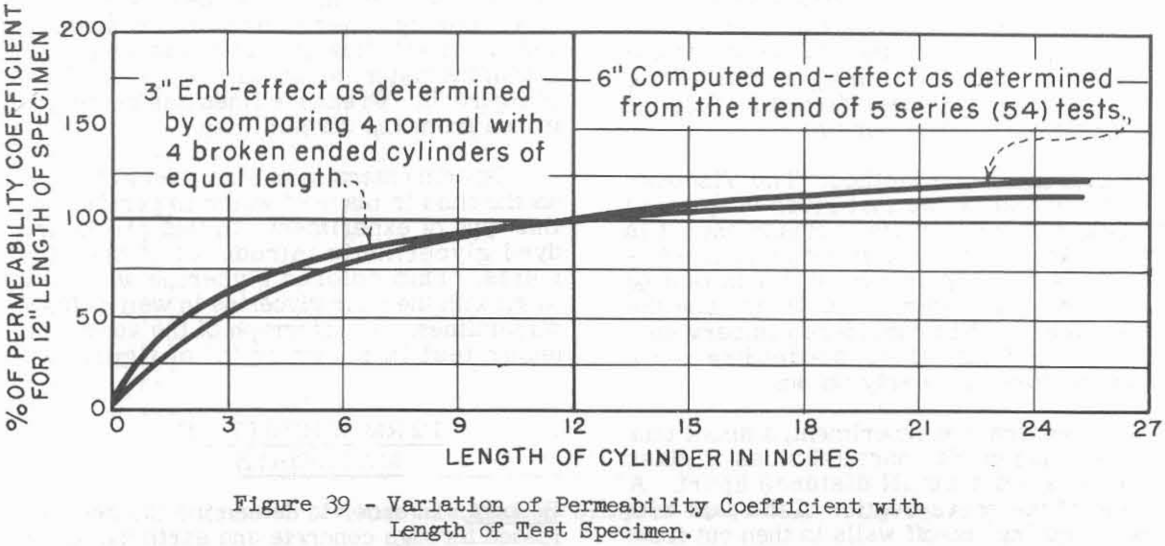


Figure 39 - Variation of Permeability Coefficient with Length of Test Specimen.

TABLE 3

TYPICAL PERMEABILITY COEFFICIENTS FOR VARIOUS MATERIALS

(Coefficient represents quantity of water in cubic feet per second, per square foot of surface exposed to percolation, passing through one foot of material with one foot of head differential $Q = \frac{K\ H}{L}$.)

Materials	$K \times 10^{12}$	
Granite specimen	2 -	10
Slate specimen	3 -	7
CONCRETE and MORTAR, w/c = 0.5 to 0.6	1 -	300
Breccia specimen	20 -	
CONCRETE and MORTAR, w/c = 0.6 to 0.7	10 -	650
Calcite specimen	20 -	400
CONCRETE and MORTAR, w/c = 0.7 to 0.8	30 -	1,400
Limestone specimen	30 -	50,000
CONCRETE and MORTAR, w/c = 0.8 to 1.0	150 -	2,500
Dolomite specimen	200 -	500
CONCRETE and MORTAR, w/c = 1.2 to 2.0	1,000 -	70,000
Biotite gneiss in place, field test	1,000 -	100,000
Sandstone specimen	7,000 -	500,000
Cores for earth dams	1,000 -	1,000,000
Slate in place, field test	10,000 -	1,000,000
Face brick	100,000 -	1,000,000
CONCRETE, unreinforced canal linings, field test	100,000 -	2,000,000
*Steel sheet-piling, junction open 1/1,000 inch with 1/2 inch of contact and 18-inch sections	500,000 -	
*CONCRETE, restrained slabs with 1/4 percent to 1/2 percent reinforcing--30° temperature change	1,000,000 -	5,000,000
Water-bearing sands		1,000,000,000

*Flow through 1-foot length of crack, 1 foot deep, $Q = 30{,}000 \frac{H}{L} b^3$, where b represents the crack width in feet.

about 1 to 100. For a given water-cement ratio with aggregate ranging from 1/4 to 9 inches maximum size, the average range in permeability was about 1 to 30. In both of these comparisons the cement content was also a variable.

3. Analysis of the physical make-up of concrete indicates that percolating water finds passage mainly through the following:

(a) Inter-sand voids above the settled cement paste, the size of the voids increasing rapidly with water-cement ratios over 0.4 to 0.5 by weight.

(b) Relatively minute voids in the cement paste, the porosity of the paste and the size of the voids depending more on the state of chemical reaction than on the water-cement ratio.

(c) Voids underneath the larger aggregates, caused by the settlement of mortar and paste, the amount of voids depending mainly on the size of the aggregate and the water-cement ratio.

4. Increasing the age of the test specimen, without interruption in curing, to the time when the permeability coefficient was determined caused a relative reduction in permeability in the ratio of 3 to 1 between the ages of 20 and 60 days, respectively, and in the ratio of 2 to 1 between the ages of 60 and 180 days, respectively. The extent to which percolating water may be expected to compensate for interrupted moist-curing was not established in the tests completed to date.

5. Due either to the manner of preparing specimens for test or to other causes, there was an end-effect which made short specimens less permeable per unit of length than long specimens. The end-effect was found

TABLE 4

REPRESENTATIVE VALUES OF THE PERMEABILITY COEFFICIENT, K

Group	Material	K, feet per year	K, centimeters per second	Class
1	Concrete (water-cement ratio = 0.5 to 0.6)	.000,024 to .000,36	.000,000,000,023 to .000,000,000,35	11-10
	Granite	.000,048 to .000,24	.000,000,000,046 to .000,000,000,23	11-10
	Concrete (w/c = 0.6 to 0.7)	.000,072 to .002,4	.000,000,000,07 to .000,000,002,3	11-9
	Slate	.000,072 to .000,17	.000,000,000,07 to .000,000,000,16	11-10
	Concrete (w/c = 0.7 to 0.8)	.000,19 to :014,2	.000,000,000,18 to .000,000,013,7	10-8
	Breccia	.000,48 on up	.000,000,000,46 to	10-
	Calcite	.000,72 to .096	.000,000,000,7 to .000,000,093	10-8
	Limestone	.000,72 to .12	.000,000,000,7 to .000,000,12	10-7
	Concrete (w/c = 0.8 to 1.0)	.000,96 to .096	.000,000,000,93 to .000,000,093	10-8
	Limestone as at Madden Dam	.001,2 to .096	.000,000,001,2 to .000,000,093	9-8
	Dolomite	.004,8 to .012	.000,000,004,6 to .000,000,012	9-8
	Clay cores for earth dams	.024 to 24.	.000,000,023 to .000,023	8-5
	Biotite gneiss, undisturbed	.024 to 2.4	.000,000,023 to .000,002,3	8-6
	Sandstone	.17 to 12.	.000,000,16 to .000,012	7-5
	Slate, undisturbed	.024 to 24.	.000,000,023 to .000,023	8-5
	Concrete Canal lining	2.4 to 48.	.000,002,3 to .000,048	6-5
	Colorado River silt	1,500.		3
2	Beach sand	9,200. to 22,300	.008,9 to .021,6	3-2
	Dune sand	19,100.	.018,5	2
	River sand	42,000. to 276,000	.041 to .266	2-1
	Loam and river sand	2,300.	.002,2	3
	Undisturbed soils, fine sand to gravel	16,900. to 327,000	.016,3 to .316	2-1
3	Casper Alcova	.46	.000,000,44	7
	Rye Patch	.50	.000,000,48	7
	Hyrum	.41	.000,000,40	7
	Agency Valley	.02	.000,000,019	8
	All-American Canal	.20	.000,000,19	7
	Belle Fourche Dam	.20	.000,000,19	7

Group 1.--Compiled by S. P. Wing and A. F. Johnson (See Tech. Memo. No. 377).
Group 2.--K. Terzaghi, in "Erdbaumechanik."
Group 3.--Average values obtained in earth laboratory.

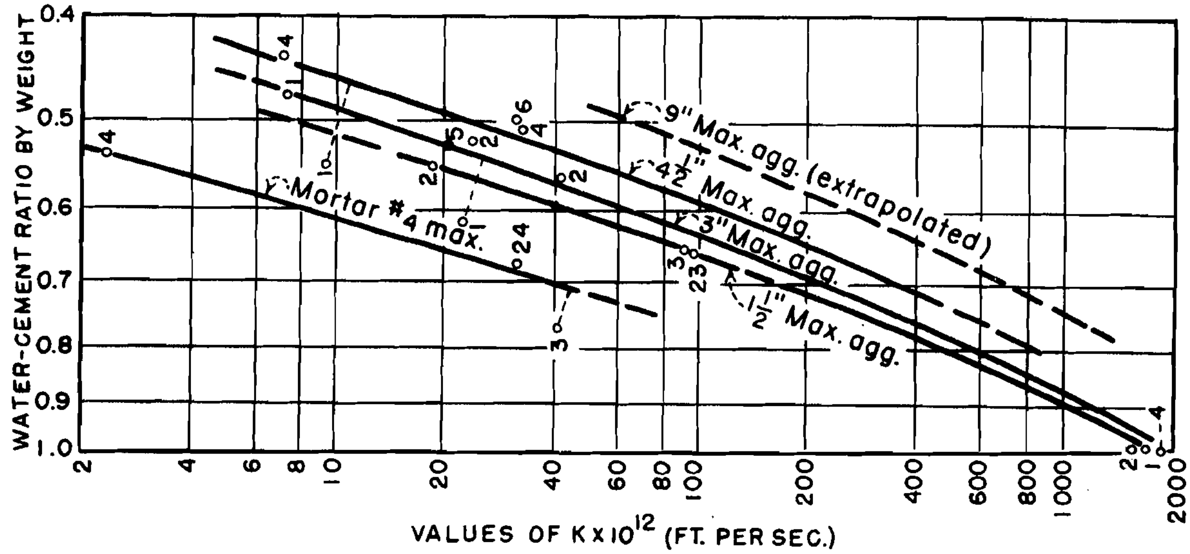


Figure 40 - Variation of Permeability Coefficient with Water-Cement Ratio and Maximum Aggregate.

to be equivalent to increasing the length of the specimen about 6 inches (or 3 inches for each end).

6. Percolation of water through concrete gradually removes the chemical compounds of the cement through solution. The amount of percolating water required to bring about a given degree of dissolution, dependent on the character of the supply water, is directly proportional to the cement content. On the premise that removal of 25 percent of the original lime content of the cement is accompanied by little strength loss of the concrete, as indicated by the relatively few tests made to date, it is estimated that at least 35 cubic feet of water, equivalent to distilled water in corrosive properties, must percolate per pound of cement before one-half of the strength of the concrete is sacrificed.

7. The reasonably satisfactory correlation of permeability test data from many sources by means of the permeability coefficient, indicates that permeability is a definite physical property of concrete susceptible of evaluation.

8. Study of the pore structure of concrete indicates the possibility of uplift acting on 85 to 95 percent of the pore area of the concrete penetrated. However, in large gravity dams the time required to develop uplift through the entire section may be many years.

Example 5. To reduce head loss and lower the leakage, a 6-inch concrete lining is proposed for a 10-foot inside-diameter tunnel two miles long under a 200-foot head, carrying snow water and located in volcanic district free of groundwater. The lining is to be placed pneumatically using a mix with 2-inch maximum aggregate and a water-cement ratio of 0.75, with one barrel of cement per cubic yard (14 pounds per cubic foot). Compute the leakage.

From figure 40, $K = 350 \times 10^{-12}$, and by figure 39 the value applicable to a 6-inch layer of concrete is 0.8 of this amount, or 280×10^{-12}. Then leakage per square foot of tunnel is

$$q = K \frac{H}{L} = \frac{280}{10^{12}} \times \frac{200}{0.5}$$

$$= \frac{112{,}000}{10^{12}} \text{ second-feet,}$$

and the total leakage from the tunnel is

$$Q = (\pi D L) \times \frac{112{,}000}{10^{12}}$$

$$= 0.04 \text{ second-feet, or } 3{,}500 \text{ cubic feet per day.}$$

Soils Permeability. The determination of the coefficient of permeability, K, for soils is difficult. Experimental methods which do not employ undisturbed samples give questionable results, for it is known that compaction affects the permeability of the soil. Laboratory tests on undisturbed samples may yield good results, but the method is expensive, applies to small regions, and sometimes samples are difficult to obtain.

Field permeability tests have obvious advantages over laboratory testing, since they more nearly approach actual flow conditions and give average results for a relatively large region. The best known and probably most reliable field tests are the Theim[8] and Theis[9] tests. These tests employ a pumping well which fully penetrates the aquifer to bedrock. Radially placed observation holes are necessary to supply test information. These tests are usually expensive and may be impossible to run, but, when possible, they yield excellent average results for a large region. They are not suitable for measuring anisotropy, local effects, or variation in the permeability of successive strata comprising the aquifer. In the Theim test, equilibrium must be approached; however, the drawdown at the well may be any percent of the depth of the aquifer provided H is treated according to equation (81), which follows. In the Theis test, equilibrium need not be achieved; here, however, the drawdown should not be more than 10 percent of the depth of the aquifer.

Theoretical investigations and field experience both indicate that field permeability tests can and should be chosen for maximum simplification of field testing procedure. Simplified field permeability tests have been developed which are inexpensive and applicable to either localized or large region testing. These tests require one uncased or partially cased hole per test and measure the outflow or inflow rate from this hole under a known constant head. Test procedure is essentially the same for all physical conditions of the material under test, and the results are obtained simply and used for permeability determination.

Use of the simplified test procedure introduces slight systematic distortion in the results as the geometrical applicability limits are approached. Field experience has shown, however, that the magnitude of other indeterminate influences such as peripheral compaction, peripheral silting, local heterogeneity, capillary action, and sometimes chemical effects, will usually make this geometrical distortion trivial by comparison.

8 Theim, A., in Forchheimer, op. cit., p. 70.

9 Theis, Charles V., "The Relation between the Lowering of the Piezometric Surface and the Rate and Duration of Discharge of a Well Using Ground-Water Storage," Transactions American Geophysical Union, 16th Annual Meeting, April, 1935.

A field permeability test consists essentially of an artificially induced seepage flow system with known boundary conditions and flow quantities. If a flow function ϕ can be found which satisfies the installed field boundary conditions, and equations (11), (13), or (16), the soils permeability can be computed directly. A logical procedure is to induce a simple flow system for which ϕ is known. Systems such as rectilinear flow, radial flow, or spherical flow are simple systems to install and are easily computed.

These three flow systems will be discussed separately to indicate more clearly their zones of application.

Rectilinear Flow. Rectilinear flow, equation (1), was used by Darcy in his original observations associated with sand filter beds. This type of flow results from allowing water to percolate through a volume of undisturbed soil surrounded by an impervious cylinder. The standard laboratory permeability test uses rectilinear flow and should give excellent results if:

1. The material is not disturbed.

2. Leakage or excessive flow at the cylinder-material interface is eliminated.

3. Gradients through the sample give velocities which are within the valid region for Darcy's law.

Rectilinear flow is rarely used in field permeability tests because of the mechanical difficulties involved and the limited region tested. Its chief application is in the measurement of anisotropy.

Two-Dimensional Radial Flow. Equations (20) through (26) give the mathematical development for two-dimensional radial flow. Equation (24) states that the quantity of flow from or to a well that fully penetrates a confined homogeneous pervious stratum is directly proportional to the depth, $T = z$, of the stratum; the permeability, K, of the stratum; and the differential head, $H = p_b - p_a$ (see figure 2); but is inversely proportional to the logarithm of the ratio of radius b to radius a. It is evident that flow to a well which fully penetrates a horizontal aquifer is two-dimensional radial flow. When the pervious bed has no confining upper impervious stratum (nonartesian), flow to or from a well will have an axial component in the vicinity of the well. This variation from the idealized flow has a negligible effect on the results if H is small relative to T. The percent error will be of the same order as H to T. When H is large relative to T, Muskat [10] has shown that equation (24) may be applied to a radial gravity flow system if the driving head H is modified according to the equation

$$H_{(gravity)} = \frac{2TH - H^2}{2T} \quad .. (81)$$

Equation (81) defines a modified driving head which approaches H/2 as H approaches T and is almost equal to H as H becomes small relative to T.

Thus, a two-dimensional radial flow system is readily reproduced and analyzed if

1. Full penetration of the pervious layer by the pumping well can be achieved.

2. Observation wells are available for determining the driving head between successive radii.

The full penetration requirement makes the test cost excessive for deep pervious strata. However, the tests are ideally suited to relatively thin pervious strata already equipped with pumping or drainage wells where only the horizontal permeability is desired.

Three-Dimensional Radial Flow.

a. Saturated Material. Equations (54) through (57) give the mathematical development for three-dimensional radial flow. Equation (57) states that the quantity of flow from or to a spherical source or sink is directly proportional to the driving head, H, ($\emptyset_b$ - $\emptyset_a$); the permeability of the surrounding material; and the radius of the source or sink. It is inversely proportional to one minus the ratio between the inner and outer radii used in measuring the differential head. The effect of the ratio of radii in the denominator is negligible if b is large relative to a. In most real cases b will be at least 20a, therefore, neglecting this term will change the results by about 5 percent or less. For determining the field permeability of soils, only hemispherical flow need be considered, and letting the outer radius b be large compared to radius a leads to the simplified form of equation (57)

$$Q = 2\pi HKa \quad \ldots\ldots\ldots\ldots (82)$$

where

$$H = \emptyset_b - \emptyset_a.$$

This simplification immediately eliminates the need for an observation well if the ground water level is known and if no major obstacle, such as an impervious layer, is closer than 5a to the source.

It is not necessary to make the test well for a three-dimensional radial flow system fully cased except for a hemispherical open end. Most test wells have cylindrical active lengths that are either screened, perforated, or uncased. To permit use of this type of test well, conductivity coefficients C_s, which give the equivalent hemispherical radius of a cylindrical well, have been determined. These dimensionless coefficients have been plotted against the ratio of cylinder length L_A to radius r_1 in figure 41. For a perforated cylindrical well, an effective well radius was found to be

$$r_{1(effective)} = r_1 \left(\frac{\text{area of perforations}}{\text{cylinder wall area}}\right)$$

In applying this result to a well with closed bottom, the effective C_s was that obtained from the curve on figure 41 at

$$L_s/r_{1(effective)}.$$

For an open bottom well,

$$4\left[r_1/r_{1\,(effective)}\right]$$

should be added to the above. The effective hemispherical radius can be computed directly from these coefficients as

$$r_{(effective)} = \frac{C_s r_1}{2\pi} \quad \ldots\ldots\ (83)$$

The effective radius will always lie numerically between the cylinder length and the radius length. These conductivity coefficients were obtained from experimental and analytical results. The experimental results include field, sand model, and electric analogy values. The analytical results include a solution for partly penetrating cylindrical wells by Muskat [11] and a solution by F. E. Cornwell (see Appendix A) for flow from a cylindrical element to a plane potential surface. Cornwell's solution

[10] Muskat, op. cit.

[11] Muskat, op. cit.

yields a very simple expression for the conductivity coefficient

$$C = \frac{L_A}{r_1} \frac{2\pi}{\ln \frac{L_A}{r_1}} \quad \ldots\ldots\ldots\ldots (84)$$

which fits the curve of figure 41 very well for values of $L_A \geqq 20r_1$, and shows that the shape of the outer boundary of the system is relatively unimportant in most three-dimensional flow systems. It should be pointed out here that equation (81) may be used to modify the head for gravity effects in three-dimensional flow as well as two-dimensional flow.

In some test areas, insufficient geological information may be available to define the boundaries of the pervious material. It is then impossible to decide whether two or three-dimensional radial flow is more nearly applicable to the problem. Solutions for both assumed ideal cases will yield values which define the limits between which the average permeability must lie. The value based on two-dimensional radial flow will always give the upper limit and, in horizontally arranged layers of material, will usually give conservative adequate results. The ratio of limiting permeabilities will be

$$\frac{K_{(2\text{-dim})}}{K_{(3\text{-dim})}} = \frac{\ln \frac{r_2}{r_1}}{\ln \frac{L_A}{r_1}} \quad \ldots\ldots\ldots (85)$$

b. Unsaturated Material. Three-dimensional radial flow from a cylindrical well in an unsaturated isotropic pervious bed requires some special treatment. R. E. Glover (see Appendix B) has developed a precise solution for the steady-state flow from a well into an infinite unsaturated medium. This solution is based on flow from an array of point sources in a uniform stream. The relation between Q, h_1, r_1, and K was found to be

$$K = \frac{Q}{2\pi h_1^2} \left[\sinh^{-1} \left(\frac{h_1}{r_1}\right) - 1 \right] \ldots (86)$$

where

h_1 = the depth of the water in the test well.

All of the developments given here have been applied to partially penetrating wells and to partly cased wells. Therefore, different limits of integration were applied to Glover's solution to yield the more general expression

$$K = \frac{Q}{2\pi(2Ah_1 - A^2)} \left[\sinh^{-1}\left(\frac{L_A}{r_1}\right) - \left(\frac{L_A}{h_1}\right) \right] \quad \ldots\ldots\ldots\ldots (87)$$

To reduce the labor involved in solving this equation, a set of coefficients, C_u, for a wide range of h_1/r_1 and L_A/h_1 ratios has been computed and plotted in figure 43. These values can be used with the familiar three-dimensional radial flow relation given by equation (82) with $2\pi a$ replaced by $C_u r_1$. Thus

$$K = \frac{1}{C_u r_1} \frac{Q}{h_1} \quad \ldots\ldots\ldots\ldots (88)$$

Applications to Soils Permeability. From the preceeding discussion it can be seen that preliminary permeability investigations can be made by very simple, rapid, and inexpensive field tests. The most accurate type of investigation employs the two-dimensional radial flow systems of Theim and Theis tests. Both tests require observation wells and a pump well which penetrates the aquifer by at least 85 percent of its depth. In the Theim test steady-state conditions are required. However, the drawdown at the well may be any percent of the total depth of aquifer. In the Theis type of test steady-state conditions need not be established. Drawdowns may be measured as a function of time. However, in the Theis test (often referred to as the nonequilibrium tests) the drawdown at the well should not exceed 10 percent of the depth of the aquifer. If the extra time and expense of these two-dimensional tests are not justified, then the simple three-dimensional radial flow test may be used, and the systematic error estimated by equation (85).

Examples 6, 7, and 8 give applications useful in determining the permeability of unsaturated soils, and Examples 9, 10, and 11 may be used in determining the permeability of saturated soils under artesian effects or where the drawdown at the well is not more than 10 percent of the depth of aquifer. Problems are often encountered where the drawdown at the well may exceed the permissible 10 percent value. Where this occurs gravity flow becomes important and Examples 12, 13, and 14 are given to demonstrate the effect of combining gravity flow with radial flow in a saturated soil.

In field permeability tests the depth of well penetration into a material and the proximity of the water table or impervious boundary to the end of the well affect the flow pattern and hence need be considered in evaluating K. Figure 54 indicates the proper equations to use for determining K at successive depths of a well from unsaturated into saturated stratum.

In performing permeability tests near the ground surface, rectangular shaped test pits may be selected rather than circular shaped wells. Experiments were made to determine the effective radius, $r_{eff.}$, in terms of 1/2 the shorter side of the rectangle as a function of the aspect ratio of the rectangle. These results, given in figure 44, make it possible to use the conductivity coefficients for circular wells given in figures 41, 42, and 43.

Acknowledgements.

D. H. Jarvis, W. T. Moody, H. B. Phillips, H. J. Kahm, and I. E. Allen assisted in conducting the electric analogy tests, in the mathematical developments, and in carrying out necessary computations. Acknowledgement is also made of the excellent flow problem developments by F. E. Cornwell and R. E. Glover given in Appendices A and B.

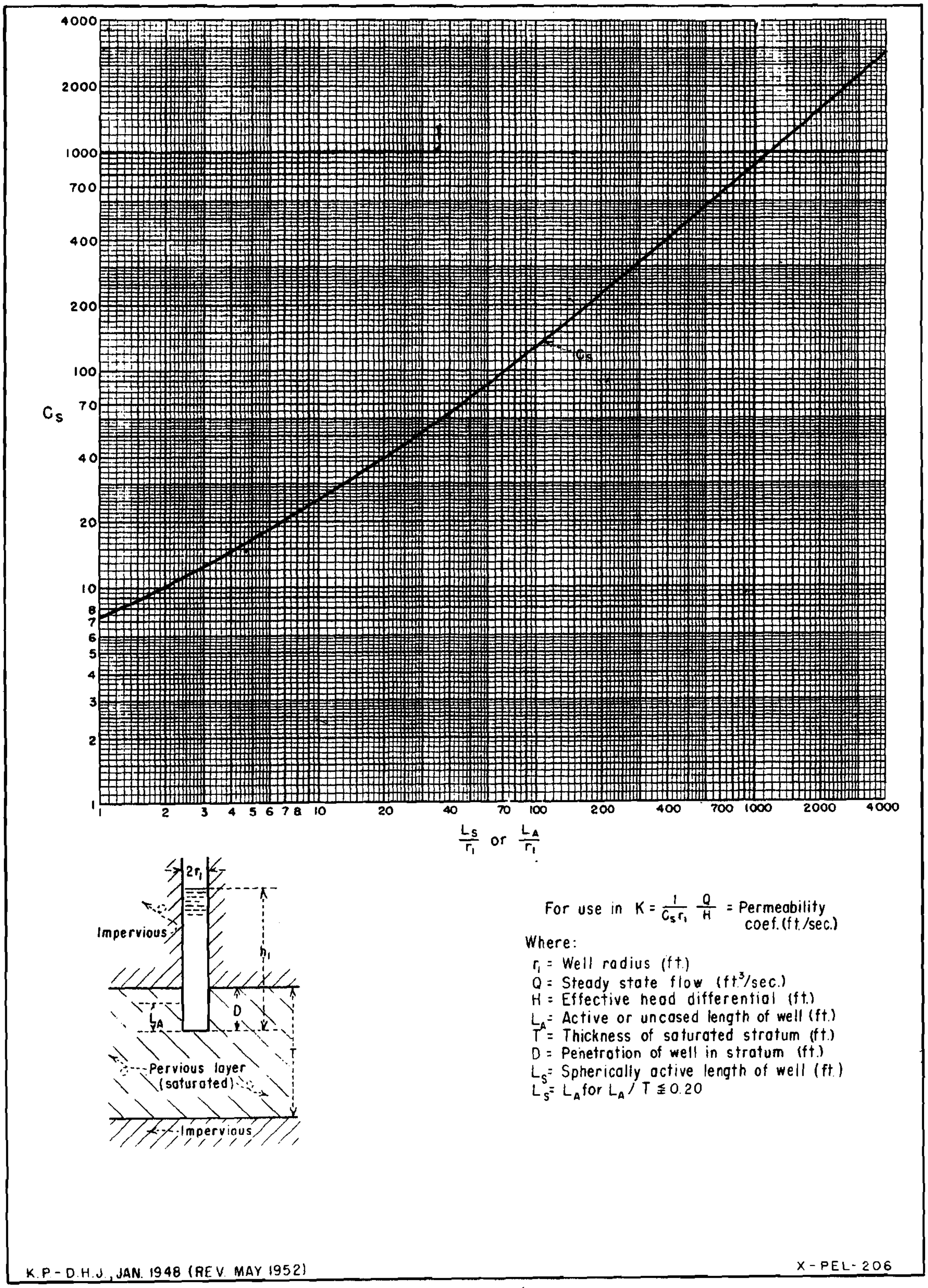


Figure 41 - Conductivity Coefficients for Semi-spherical Flow in Saturated Strata through Partially Penetrating Cylindrical Test Wells.

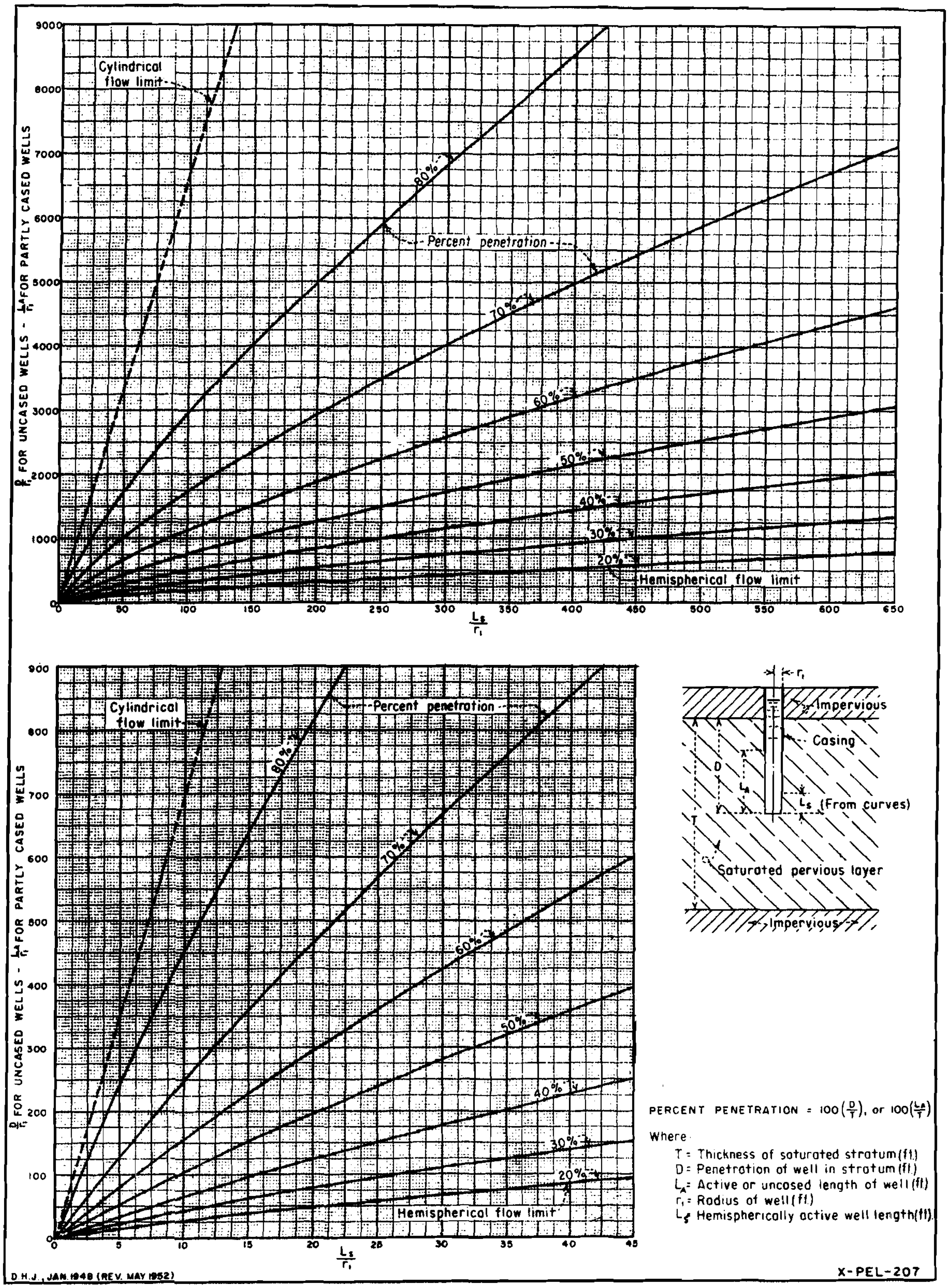
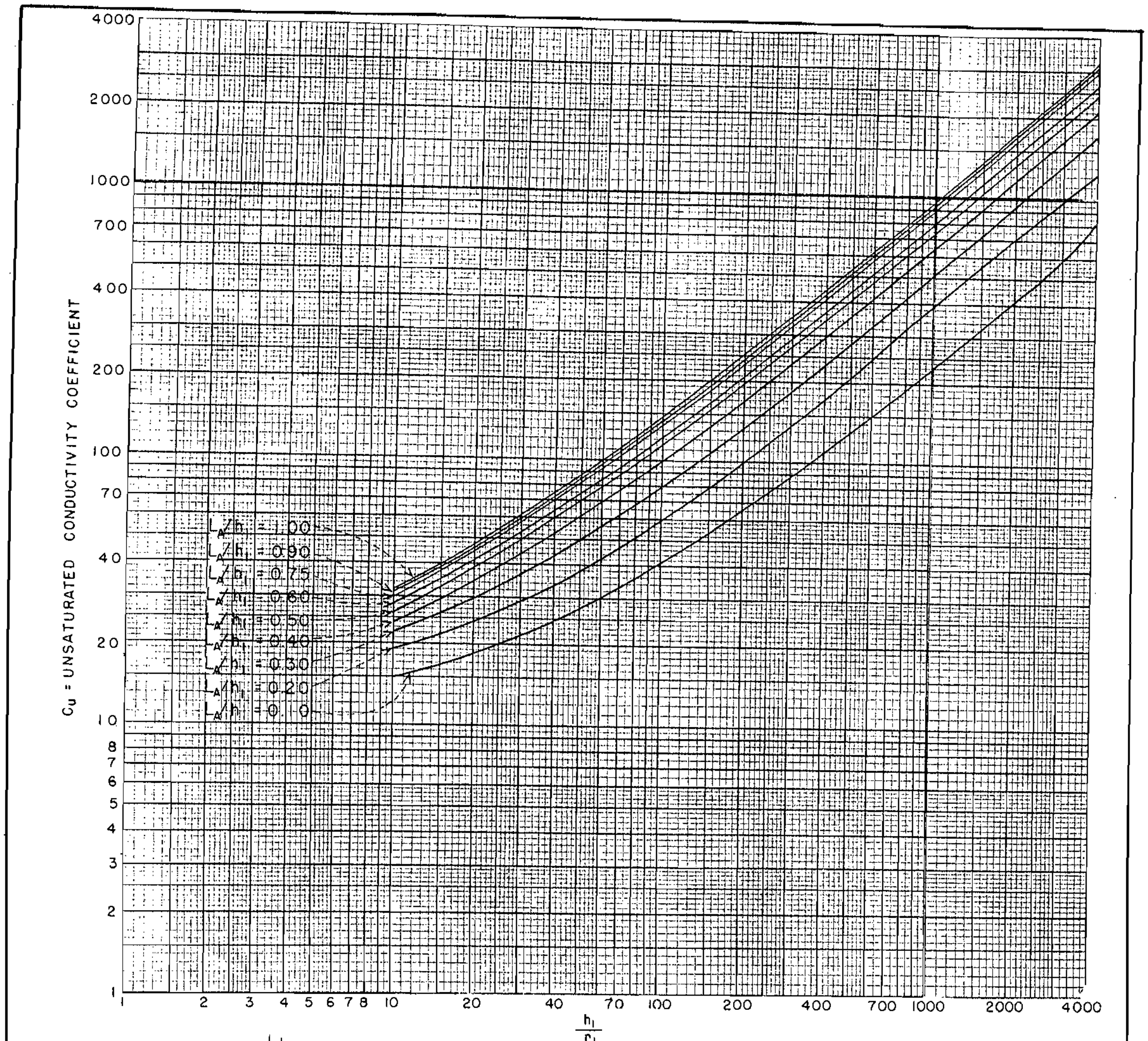
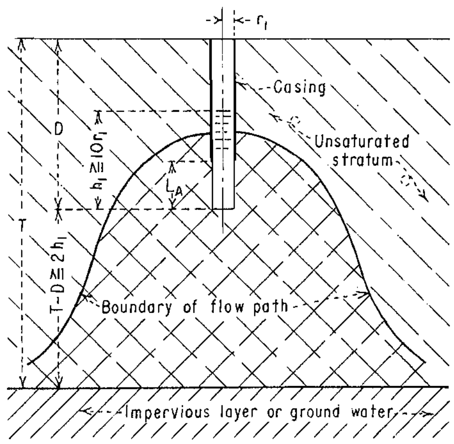


Figure 42 - Permissible Hemispherical Flow Length of Partially Penetrating Cylindrical Wells in Saturated Strata.





$$K = \frac{1}{C_u r_1} \cdot \frac{Q}{h_1} = \text{Permeability coef. (ft./sec.)}$$

Where:

- r_1 = Well radius (ft.)
- Q = Steady state flow (ft./sec.)
- h_1 = Head on well (ft.)
- L_A = Active or uncased length of well (ft.)
- T = Thickness of unsaturated stratum (ft.)
- D = Penetration of well in stratum (ft.)

APPLICABILITY LIMITS:

$h_1 \geqq 10r_1$ $\quad$ $T-D \geqq 2h_1 \geqq 20r_1$

K. P. - D. H. J., JAN. 1948 (REV. MAY 1952) X-PEL-205

Figure 43 - Conductivity Coefficients for Permeability Determination in Unsaturated Strata with Partly Penetrating Cylindrical Test Wells.

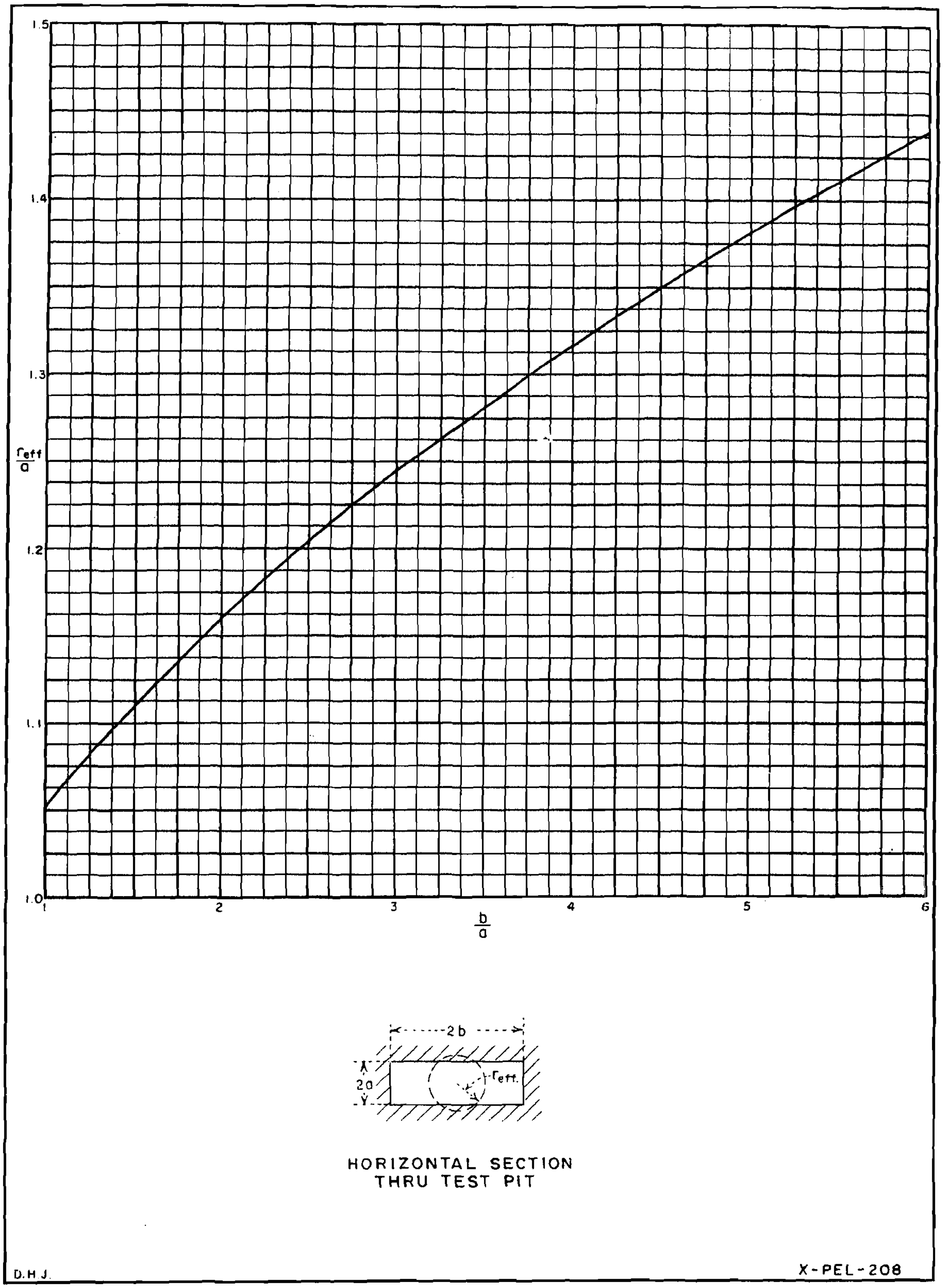
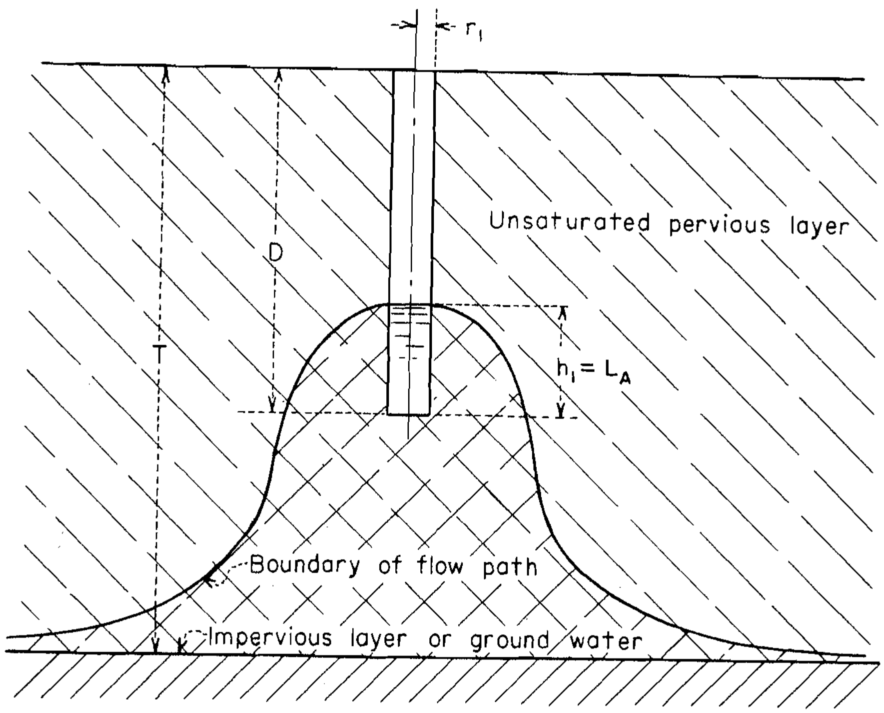


Figure 44 - Effective Cylindrical Radius of Rectangular Test Pits. (Value to be Used with Figures 41, 42, and 43 for Permeability Determination.)



FORMULA: $K = \frac{1}{C_u r_1} \frac{Q}{h_1}$

DEFINITIONS: Q = Well discharge – steady state ($ft.^3/sec.$)
C_u - From figure 43 - Use curve for $\frac{L_A}{h_1} = 1.00$
Other values as shown

NUMERICAL EXAMPLE:

Let $T = 60$ ft., $D = 35$ ft., $h_1 = 10$ ft., Then $T - D = 25$ ft. $= 2.5\, h_1$

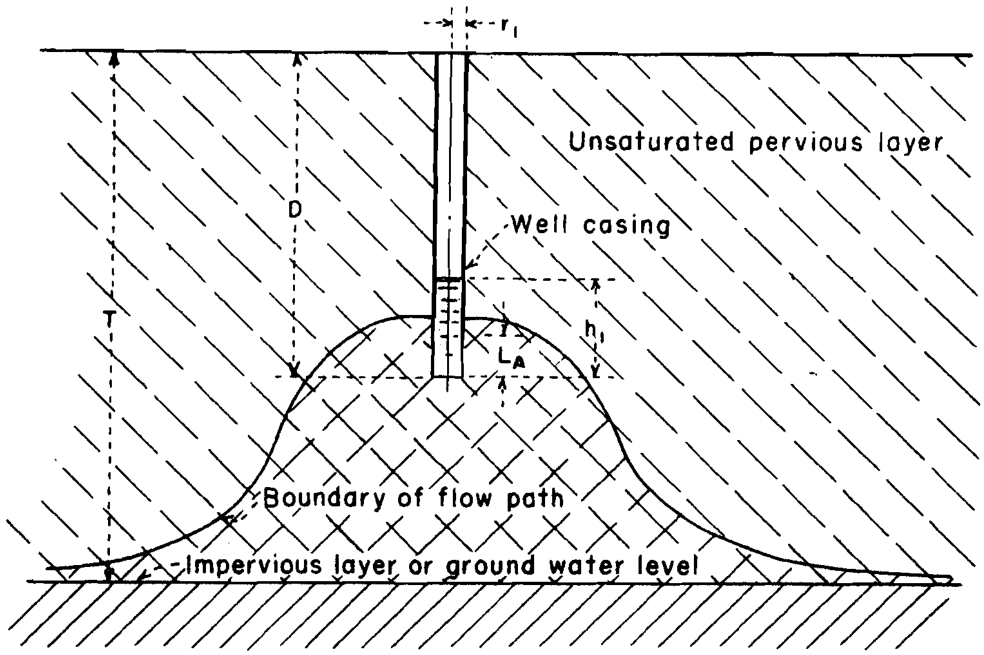
$Q = 0.10\ ft.^3/sec.$ $\qquad r_1 = 0.25$ ft.

$\frac{h_1}{r_1} = 40 \quad \therefore \quad C_u = 74.5$ (From figure 43)

$K = \frac{1}{(74.5)(0.25)} \frac{0.10}{10} = 0.00054$ ft./sec.

D.J.H., 11-26-47 (REV. MAY 1952) X – PEL – 181

Figure 45 - Example 6: Outflow from an Uncased Cylindrical Well in an Unsaturated Stratum, $T - D \geqq 2h_1$, $\frac{h_1}{r_1} \geqq 10$.



FORMULA: $K = \frac{1}{C_u r_1} \frac{Q}{h_1}$

DEFINITIONS: Q = Well discharge - steady state (ft.³/sec.)

C_u = From Figure 43 use curve for nearest $\frac{L_A}{h_1}$

NUMERICAL EXAMPLE:

Let $T = 60$ ft., $D = 35$ ft., $h_1 = 10$ ft., Then $T - D = 25$ ft. $= 2.5 h_1$

$L_A = 5$ ft., $\frac{L_A}{h_1} = 0.5$, $r_1 = 0.25$ ft., $\frac{h_1}{r_1} = 40$, $C_u = 59$

$Q = 0.10$ (ft.³/sec.)

$$K = \frac{1}{(59)(0.25)} \frac{0.10}{10} = 0.00068 \text{ ft./sec.}$$

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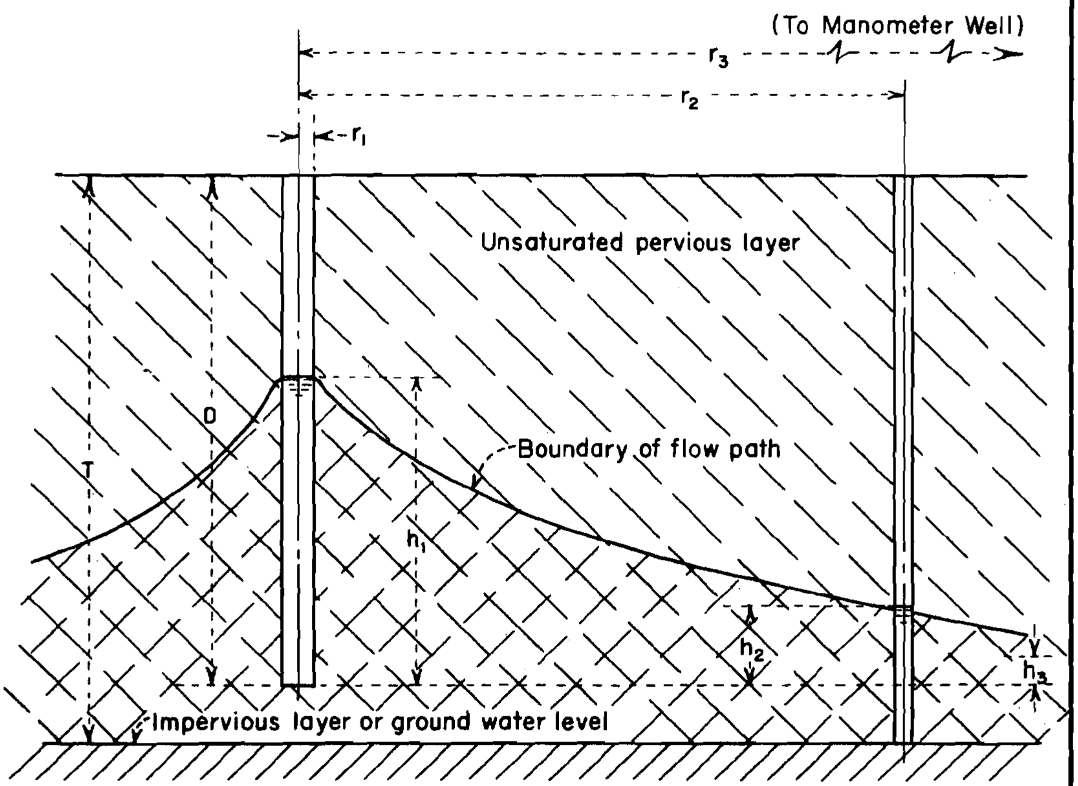
Figure 46 - Example 7: Outflow from a Partly Cased Cylindrical Well in an Unsaturated Stratum,

$$T - D \geqq 2h_1, \ \frac{h_1}{r_1} \geqq 10.$$



NECESSARY CONDITIONS:

(1) $$\frac{[35\, r_3^2\ (\Delta h_1 + 2\Delta h_3)]}{Q} \leqq 10 \quad \text{(Steady state)}$$

(2) $$r_2 \geqq \frac{h_1 + (T-D)}{2}$$

FORMULA:

$$K = \frac{\ln\left(\frac{r_3}{r_2}\right)}{\pi} \; \frac{Q}{(h_2^2 - h_3^2)}$$

DEFINITIONS:

Q = Steady state well discharge (ft.^{3}/sec.)

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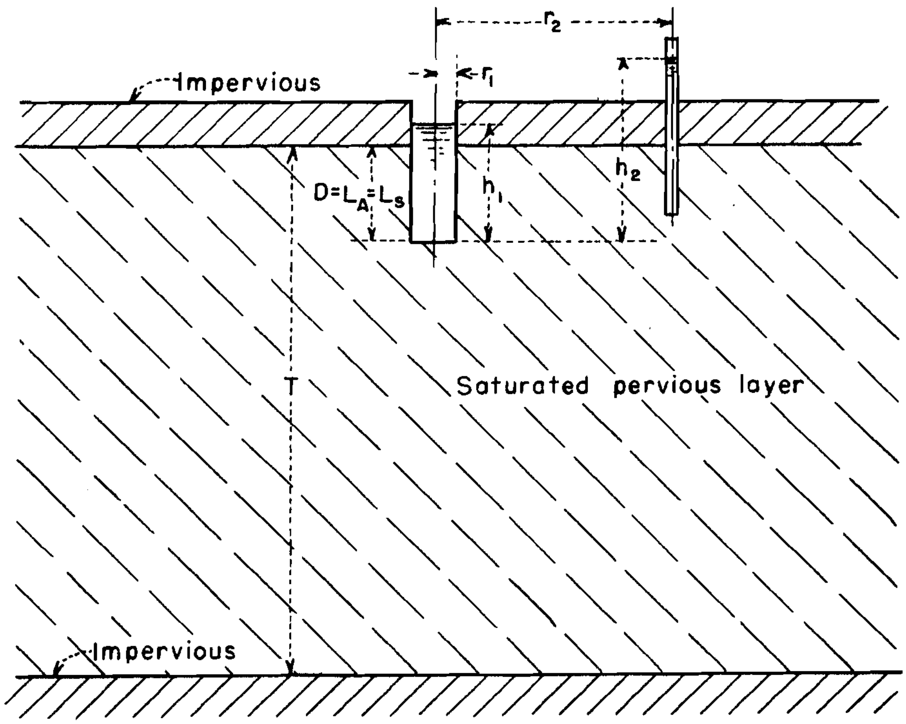
Figure 47 - Example 8: Outflow from a Cylindrical Well in an Unsaturated Stratum, $T - D < 2h_1$.



FORMULA: $K = \frac{1}{C_s\, r_1}\ \frac{Q}{H}$

DEFINITIONS
Q = Well discharge – positive into well ($ft.^3/sec.$)
C_s = From figure 41, $H = h_2 - h_1$
h_2 = Undisturbed ground water level (ft.)

NUMERICAL EXAMPLE:

Let $T = 60$ ft., $D = 9$ ft., $h_1 = 12$ ft., $h_2 = 15$ ft.

$Q = 0.10\ ft.^3/sec.$ $r_1 = 0.25$ ft., Then $D/T = 0.15 < 0.2$

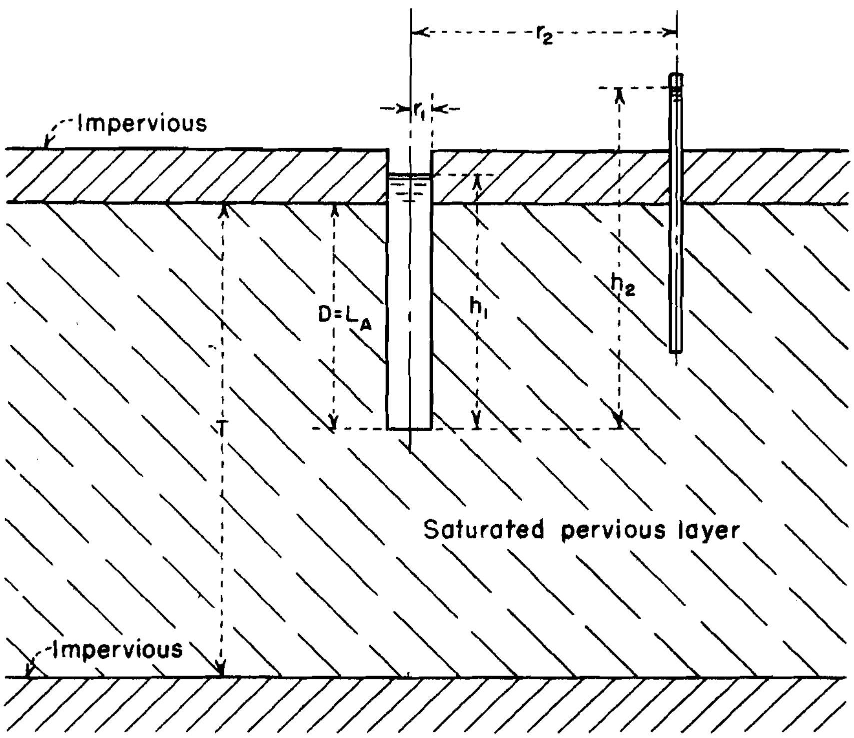
$\frac{Ls}{r_1} = \frac{D}{r_1} = 36 \quad \therefore\ C_s = 58$ (From Figure 41)

$H = 15 - 12 = 3$

$K = \frac{1}{(58)(0.25)} \cdot \frac{(0.10)}{(3)} = 0.0023$ ft./sec.

K.P., 11-28-47 (REV. MAY 1952) X-PEL-187

Figure 48 - Example 9: Inflow to an Uncased Cylindrical Well in a Saturated Stratum, $D/T \leqq 0.20$.



FORMULA: $$K = \frac{\ln \frac{r_2}{r_1}}{2\pi(D-L_s) + C_s r_1 \ln \frac{r_2}{r_1}} \cdot \frac{Q}{H}$$

DEFINITIONS: Q = Well discharge – Positive into well (ft.^{3}/sec.)

C_s = From Figure 41 and 42, $H = h_2 - h_1$ (ft.)

NUMERICAL EXAMPLE:

Let T = 60 ft., D = 30 ft., h_1 = 35 ft., h_2 = 45 ft.

Q = −0.10 ft.^{3}/sec., r_1 = 0.25 ft., r_2 = 25 ft.

Then $D/T = 0.5 \gtrless {0.20 \atop 0.85}$, $\frac{D}{r_1} = 120$, $\frac{L_s}{r_1} = 11.75$

C_s = 27.90 (Fig. 41), $D - L_s$ = 27.0625, H = 10 ft.

$$K = \frac{\ln 100}{2\pi(27.0625)+(\ln 100)(27.9)(0.25)} \cdot \frac{0.10}{10} = 0.00023 \text{ ft./sec.}$$

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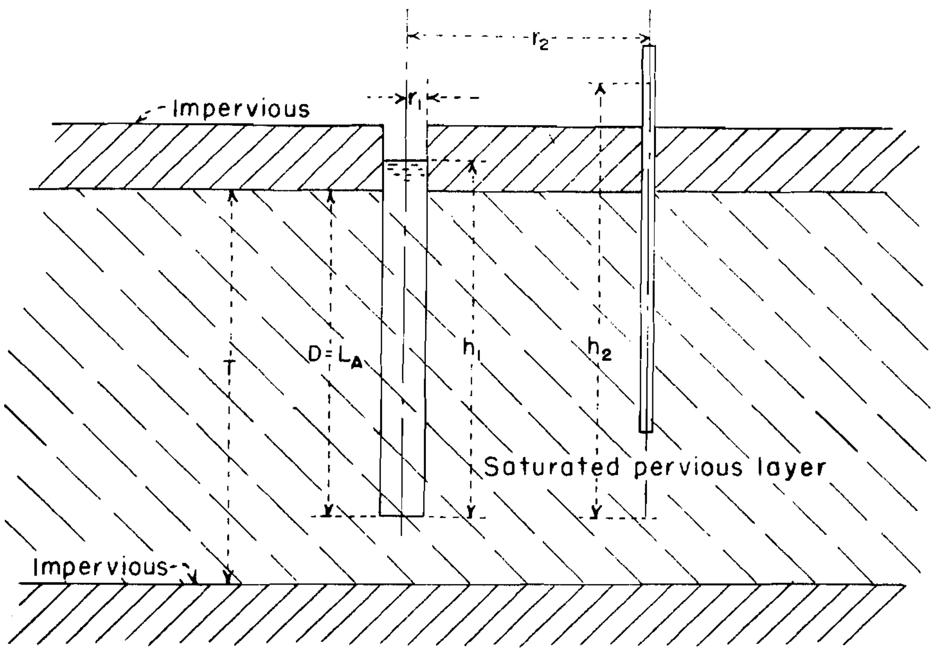
Figure 49 - **Example 10: Inflow to an Uncased Cylindrical Well in a Saturated Stratum, 0.20< D/T< 0.85.**



FORMULA: $$K = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi D} \cdot \frac{Q}{H}$$

DEFINITIONS: Q = Well discharge – positive into well (ft.^{3}/sec)

$H = h_2 - h_1$ (ft.)

NUMERICAL EXAMPLE:

Let T = 60 ft., D = 54 ft., h_1 = 60 ft., h_2 = 65 ft.

Q = +0.10 cu. ft./sec., r_1 = 0.25 ft., r_2 = 25 ft.

Then $D/T = 0.9 > 0.85$

$H = 65 - 60 = 5$ ft., $\frac{r_2}{r_1} = 100$

$$K = \frac{\ln 100}{2\pi(54)} \cdot \frac{(0.10)}{(5)} = 0.00027 \text{ ft/sec.}$$

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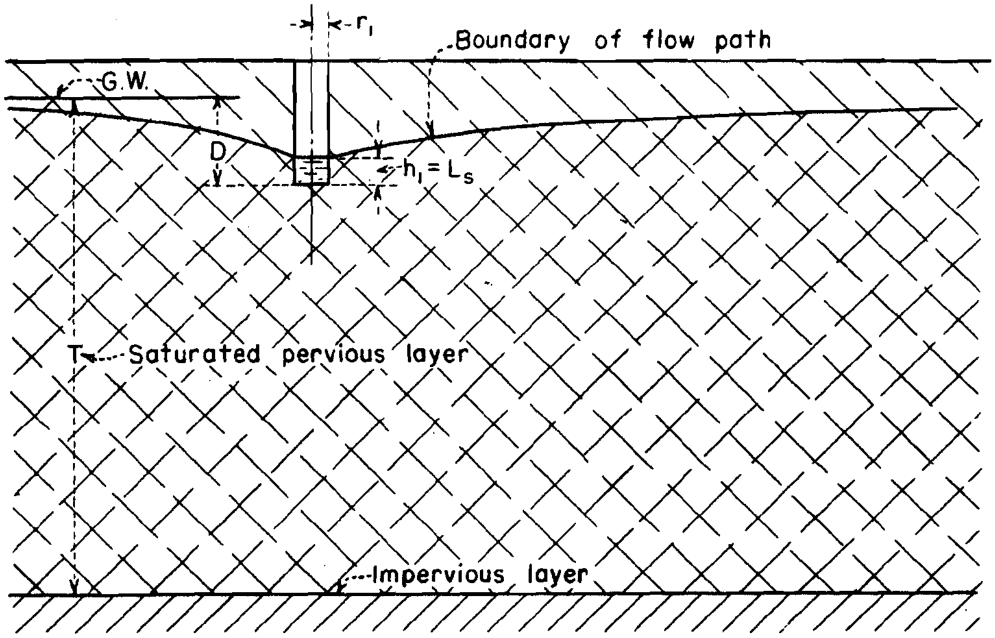
Figure 50 - **Example 11**: Inflow to an Uncased Cylindrical Well in a Saturated Stratum, $D/T \geq 0.85$.



FORMULA: $K = \frac{1}{C_s r_1} \; \frac{Q}{H}$

DEFINITIONS: Q = Well discharge - positive into well (ft.^{3}/sec.)

C_s = From figure 41 - use $\frac{h_1}{r_1} = \frac{L_s}{r_1}$

$H = \frac{D^2 - h_1^2}{2D}$ (ft.)

Other values as shown

NUMERICAL EXAMPLE:

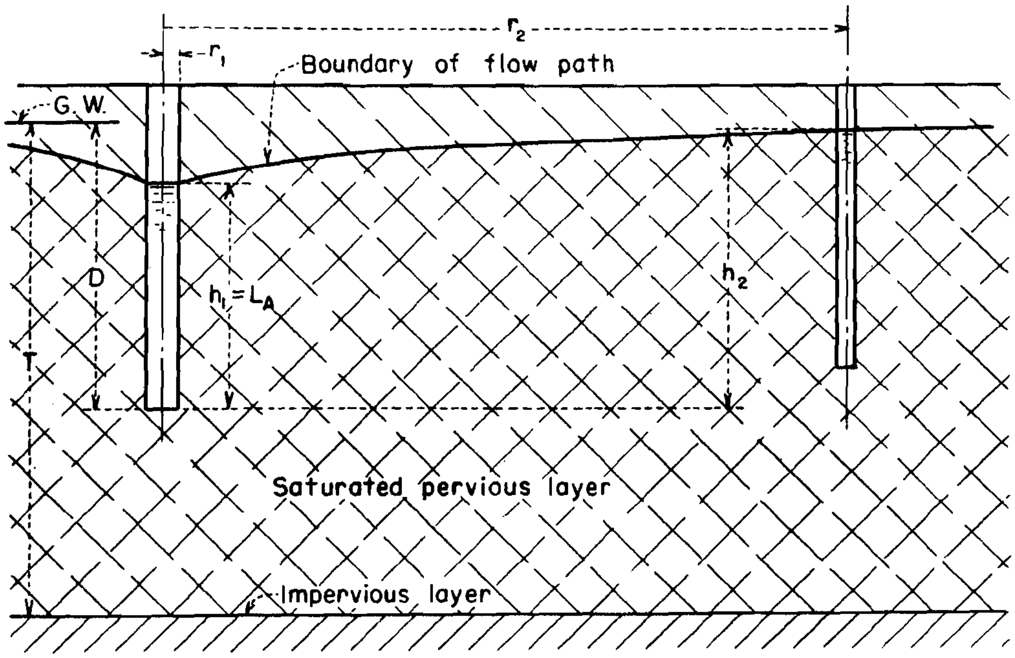
Let $T = 100$ ft., $D = 20$ ft., $h_1 = 10$ ft., $r_1 = 0.25$ ft., $Q = 0.10$ ft.^{3}/sec.

Then $\frac{h_1}{r_1} = 40$, $C_s = 63$ (From figure 41), $H = 7.5$

$$K = \frac{1}{(63)(0.25)} \cdot \frac{0.10}{7.5} = 0.00085 \text{ ft./sec.}$$

D. H. J., 12-1-47 (REV. MAY 1952) X-PEL-193

Figure 51 - Example 12: Inflow to a Partly Penetrating Cylindrical Well in a Saturated Stratum under Gravity Head, D/T ≦ 0.20.



FORMULA: $$K = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi(D - L_s) + C_s r_1 \ln\left(\frac{r_2}{r_1}\right)} \; \frac{Q}{H}$$

DEFINITIONS:

Q = Well discharge - positive into well ($ft.^3/sec.$)

L_s = From Figure 42

C_s = From Figure 41

$H = \dfrac{h_2^2 - h_1^2}{2D}$ (ft)

NUMERICAL EXAMPLE:

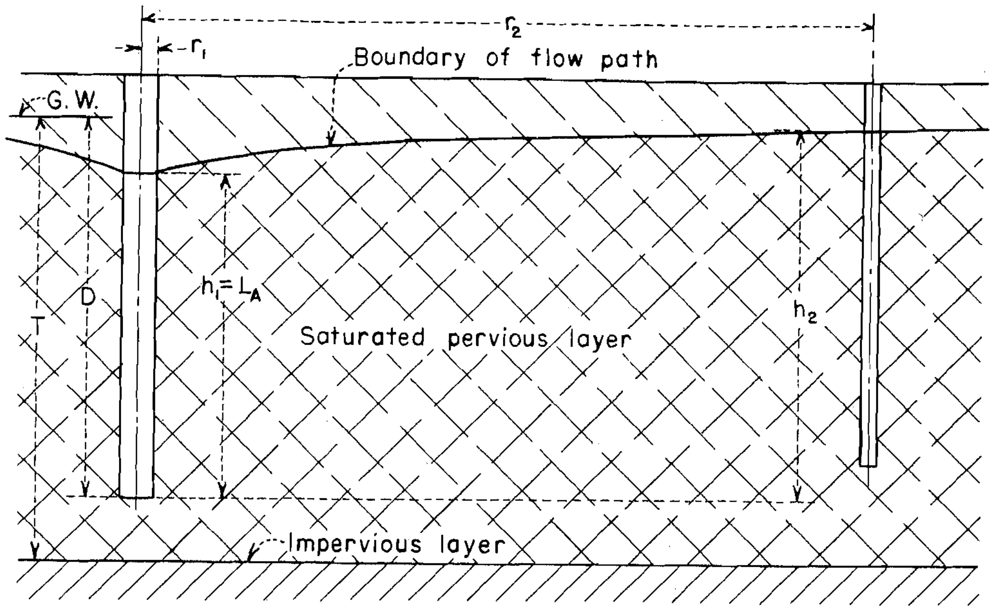
Let $Q = 0.40$ $ft.^3/sec.$, $T = 100$ ft., $D = 50$ ft., $h_1 = 30$ ft., $h_2 = 45$ ft., $r_1 = 0.25$ ft., $r_2 = 100$ ft.

Then $\frac{D}{r_1} = 200$, $\frac{L_s}{r_1} = 20.3$ (Figure 42), $C_s = 38$ (Figure 41), $L_s = 5$, $(h_1 - L_s) = 25$, $H = 12.5$, $\frac{r_2}{r_1} = 400$, $\ln\left(\frac{r_2}{r_1}\right) = 5.99$

$$K = \frac{5.99}{2\pi(25) + 38(0.25)(5.99)} \; \frac{0.40}{12.5} = 0.00090 \text{ ft./sec.}$$

D.H.J., 12-1-47 (REV. MAY 1952) X-PEL-194

Figure 52 - Example 13: Inflow to a Partly Penetrating Cylindrical Well in a Saturated Stratum under Gravity Head, $0.20 < D/T < 0.85$.



FORMULA: $K = \dfrac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi D}\,\dfrac{Q}{H}$

DEFINITIONS: Q = Well discharge - positive into well (ft.^{3}/sec.)

$H = \dfrac{h_2^2 - h_1^2}{2D}$ (ft.)

Other values as shown

NUMERICAL EXAMPLE:

Let $Q = 0.50$ ft.^{3}/sec. (Steady state)

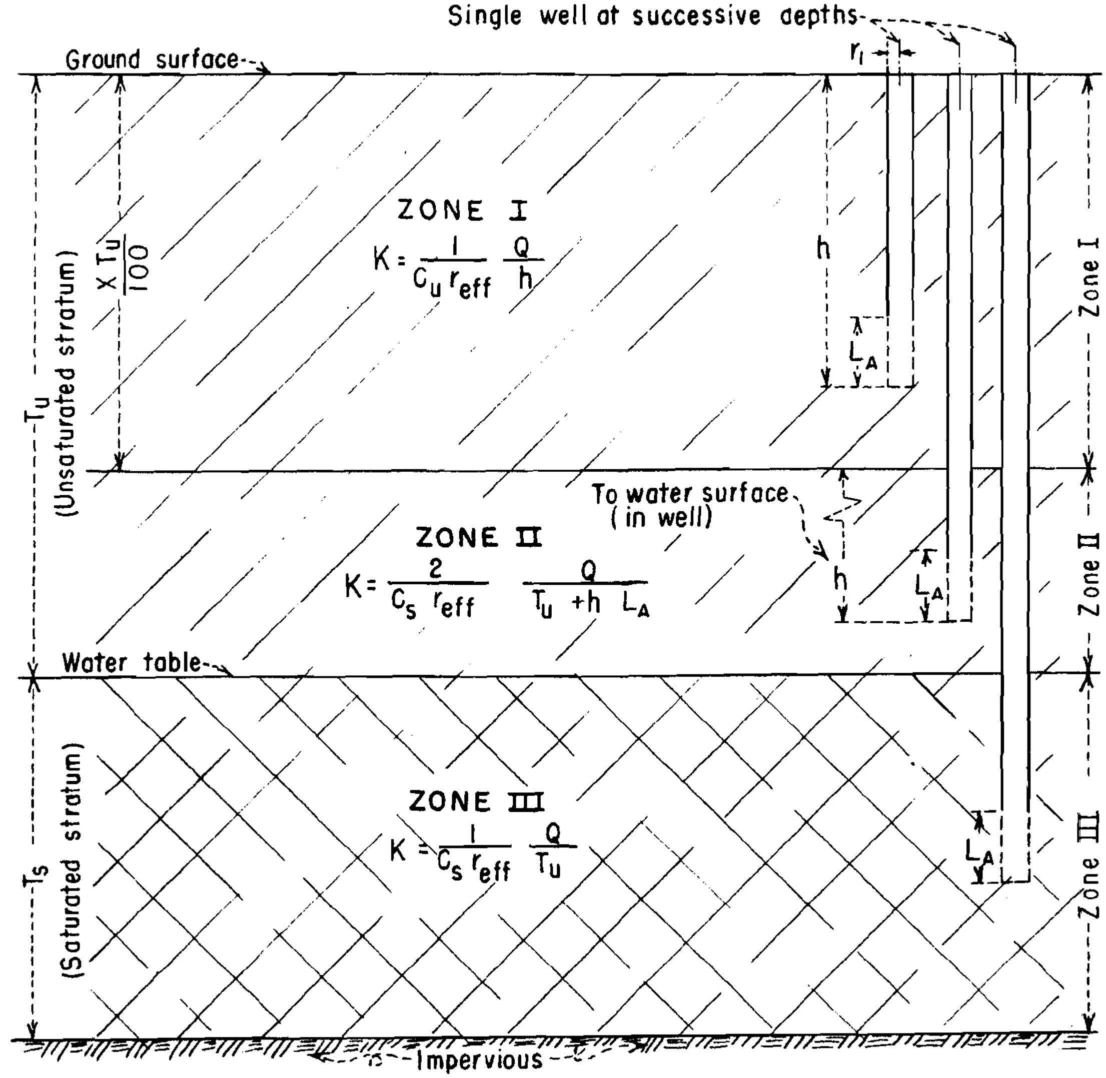
$T = 100$ ft., $D = 90$ ft., $h_1 = 70$ ft., $h_2 = 85$ ft., $r_1 = 0.25$ ft., $r_2 = 150$ ft.

Then $\dfrac{r_2}{r_1} = 600$, $\ln\left(\frac{r_2}{r_1}\right) = 6.397$, $H = 13.68$ ft.

$K = \dfrac{6.397}{(6.2832)(90)}\,\dfrac{0.50}{13.68} = 0.00041$ ft./sec.

D.H.J., 12-1-47 (REV. MAY 1952) X-PEL-195

Figure 53 - Example 14: Inflow to a Partly Penetrating Cylindrical Well in a Saturated Stratum under Gravity Head, $D/T \geqq 0.85$.



NOTATION:

K=Permeability coefficient (ft./sec.)
Q=Steady flow into well (ft.^{3}/sec.)
h=Height of water in well (ft.)
x=Percent of T_u in zone I (See Figure 55)
L_A=Length of perforated section (ft.)
r_{eff}=Effective radius of well = $r_1 \left(\frac{\text{Area of perforations}}{\text{Area of wall}}\right)$ (ft.)
C_U=Conductivity coefficient-unsaturated bed (See Figure 43)
C_S=Conductivity coefficient-saturated bed (see Figure 41)

LIMITATIONS: $T_s \geqq 5 L_A$ $\quad$ $L_A \geqq 10 r_{eff}$

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Figure 54 - Proposed Three-zone Program of Field Permeability Testing by Single Cased Well Pumping-in Test.

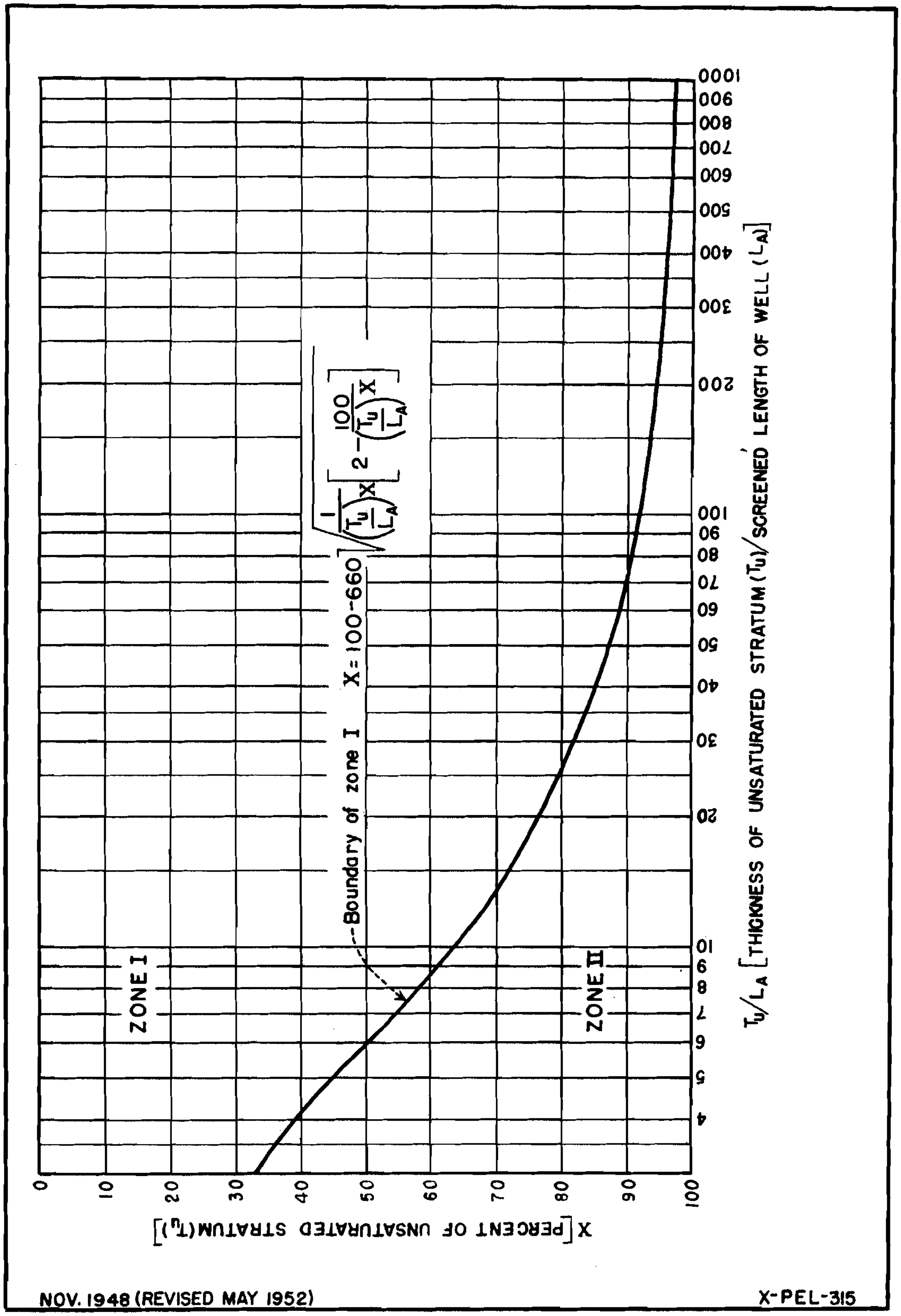

Figure 55 - Location of Zone 1 Lower Boundary for Use in Permeability Testing.

APPENDIX A

FLOW FROM A SHORT SECTION OF TEST-HOLE BELOW GROUNDWATER LEVEL

(Development by F. E. Cornwell)

This development is for the determination of the permeability coefficient, K, of a soil, by drilling a hole in the soil and permitting flow out of the well into the surrounding soil through the sides of the well only. To apply the results of the mathematical derivations to a field problem, the well in this instance must be placed in a soil that is completely saturated.

The derivation may proceed in the following manner. Let

r_1 = radius of a spherical surface concentric around the point-source,
K = permeability coefficient,
p = pressure head, and
Q = quantity of fluid flowing across any spherical shell per unit of time.

Then from Darcy's law,

$$Q = -4\pi r_1^2 \frac{\partial p}{\partial r_1} K \quad \ldots\ldots (1A)$$

and

$$\frac{\partial p}{\partial r_1} = -\frac{Q}{4\pi K r_1^2} \quad \ldots\ldots (2A)$$

By integration,

$$p = \frac{Q}{4\pi K r_1} + p_o \quad \ldots\ldots (3A)$$

Place a point-source at $y = h$ and $x = 0$.

Then

$$r_1^2 = x^2 + (y-h)^2 \quad \ldots\ldots (4A)$$

and

$$p = \frac{Q}{4\pi K\sqrt{x^2+(y-h)^2}} + p_o \quad \ldots\ldots (5A)$$

Place a source of strength Q at $y = h$, $x = 0$, and place a sink of strength $-Q$ at $y = -h$, $x = 0$. Then

$$p = \frac{Q}{4\pi K\sqrt{x^2+(y-h)^2}} - \frac{Q}{4\pi K\sqrt{x^2+(y+h)^2}} \quad \ldots (6A)$$

$$\frac{\partial p}{\partial y} = -\frac{Q}{4\pi K}\left[\frac{y-h}{\left(\sqrt{x^2+(y-h)^2}\right)^3} - \frac{y+h}{\left(\sqrt{x^2+(y+h)^2}\right)^3}\right] \quad \ldots (7A)$$

$$\left[\frac{\partial p}{\partial y}\right]_{y=0} = \frac{Q}{4\pi K}\left[\frac{h}{\left(\sqrt{x^2+h^2}\right)^3} + \frac{h}{\left(\sqrt{x^2+h^2}\right)^3}\right]$$

$$= \frac{Qh}{2\pi K\left(\sqrt{x^2+h^2}\right)^3} \quad \ldots (8A)$$

As a check, the total flow across the surface $y = 0$ is,

$$\int_0^\infty 2\pi x K \left[\frac{\partial p}{\partial y}\right]_{y=0} dx$$

$$= Q h \int_0^\infty \frac{x\,dx}{(x^2+h^2)^{3/2}}$$

$$= - Q h \left[\left(\frac{1}{x^2 + h^2} \right)^{1/2} \right]_o^{\infty}$$

$$= Q \quad \dots\dots\dots\dots\dots\dots \text{(9A)}$$

Consider a length dh of strength dq. Then $dq = q\,dh$, and

$$dp = \frac{q}{4\pi K} \left[\frac{dh}{\sqrt{x^2 + (y-h)^2}} - \frac{dh}{\sqrt{x^2 + (y+h)^2}} \right] \dots \text{(10A)}$$

Now refer to figure 56. For a line-source from $y = b$ to $y = c$ and a line-sink from $y = -b$ to $y = -c$, integrate from $h = b$ to $h = c$, where $c > b$. Then

$$p = \frac{q}{4\pi K} \left[\int_{h=b}^{h=c} \frac{dh}{\sqrt{x^2 + (y-h)^2}} - \int_{h=b}^{h=c} \frac{dh}{\sqrt{x^2 + (y+h)^2}} \right] \dots \text{(11A)}$$

For the first integral, let

$y - h = z$

$dh = -dz$

$h = b, z = y - b$

$h = c, z = y - c$

For the second integral, let

$y + h = w$

$dh = dw$

$h = b, w = y + b$

$h = c, w = y + c$

Then

$$p = -\frac{q}{4\pi K} \left\{ \left[\ln\left(z + \sqrt{x^2 + z^2}\right) \right]_{z=y-b}^{z=y-c} + \left[\ln\left(w + \sqrt{x^2 + w^2}\right) \right]_{w=y+b}^{w=y+c} \right\} \dots \text{(12A)}$$

which can be simplified and written,

$$p = -\frac{q}{4\pi K} \left[\ln \frac{y - c + \sqrt{x^2 + (y-c)^2}}{y - b + \sqrt{x^2 + (y-b)^2}} + \ln \frac{y + c + \sqrt{x^2 + (y+c)^2}}{y + b + \sqrt{x^2 + (y+b)^2}} \right] \text{(13A)}$$

Note the following equalities:

$$- \ln \frac{y - c + \sqrt{x^2 + (y-c)^2}}{y - b + \sqrt{x^2 + (y-b)^2}}$$

$$= \ln \frac{y - b + \sqrt{x^2 + (y-b)^2}}{y - c + \sqrt{x^2 + (y-c)^2}}$$

$$= \ln \frac{c - y + \sqrt{x^2 + (c-y)^2}}{b - y + \sqrt{x^2 + (b-y)^2}}$$

For the cylinder where $x = r$, let

$x = r$

$c - b = L$

$p = H$

$$q = \frac{Q}{c - b} = \frac{Q}{L}$$

$$y = \frac{b + c}{2}$$

Then,

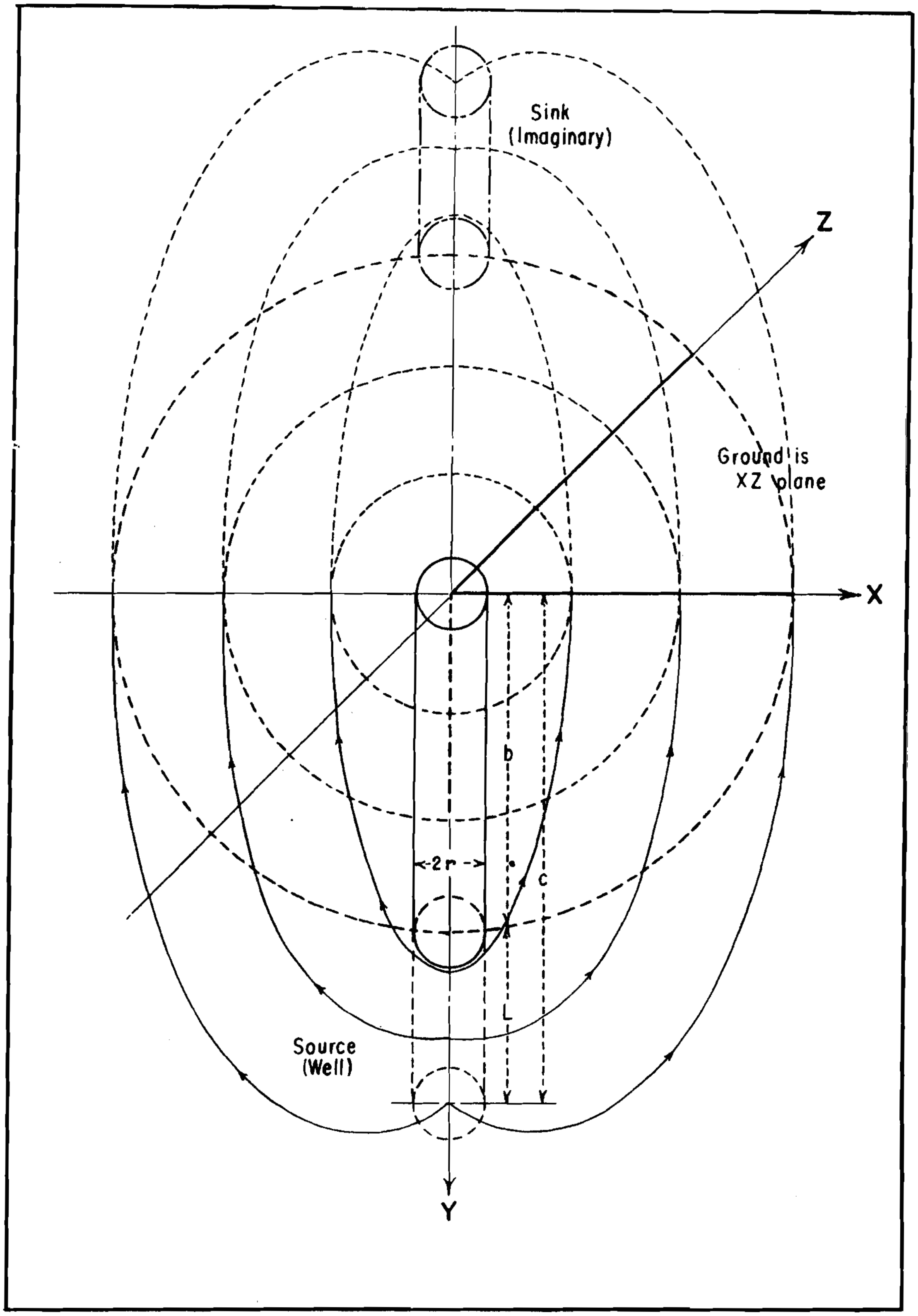

Figure 56 - Flow from Test-hole below Groundwater Level.

$$H = \frac{Q}{4 \pi K L}$$

$$\left[\ln \frac{\frac{c-b}{2} + \sqrt{r^2 + \left(\frac{c-b}{2}\right)^2}}{\frac{b-c}{2} + \sqrt{r^2 + \left(\frac{b-c}{2}\right)^2}} - \ln \frac{\frac{b+3c}{2} + \sqrt{r^2 + \left(\frac{b+3c}{2}\right)^2}}{\frac{c+3b}{2} + \sqrt{r^2 + \left(\frac{c+3b}{2}\right)^2}} \right] \quad \ldots . (14A)$$

Hence,

$$K = \frac{Q}{4 \pi L H} \left[\ln \frac{\frac{L}{2r} + \sqrt{1 + \left(\frac{L}{2r}\right)^2}}{-\frac{L}{2r} + \sqrt{1 + \left(-\frac{L}{2r}\right)^2}} \right] \quad \text{(Approx.)} \ldots . (15A)$$

or

$$K = \frac{Q}{4 \pi L H} \ln \left(\frac{L}{r}\right)^2 \text{(Approx.)} \quad (16A)$$

or

$$K = \frac{Q}{2 \pi L H} \ln \frac{L}{r} \quad \text{(Approx.)} \ldots (17A)$$

In changing from a line-source to the approximate cylindrical source, p was taken as equal to H at $x = r$ and $y = b + c/2$, which is not quite true over the remainder of length L. Also, the formula allows for some flow out of each end of the section of length L. However, neither of these approximations should be of any concern where the other limitations of application are observed; that is, where L (or c - b) is large compared with r, and c and b are each large compared with L.

Since this development is valid only for completely saturated soil, the well must be cased from the ground surface to the water table. Then H is the difference in head between water levels inside and outside of the casing. Equation (17A) may again be written in the form

$$K = \frac{Q}{H r} \cdot \frac{1}{C} \quad \ldots \ldots \ldots . (18A)$$

where

$$C = \frac{2 \pi L}{r \ln \frac{L}{r}} \quad \ldots \ldots \ldots . (19A)$$

Note here that $L = c - b$.

Values of C have been computed by equation (19A) for various L/r ratios and are shown in Table 5. The table permits an immediate determination of K in the field merely by measuring H and the corresponding Q. Then by selecting the proper value of C, K is easily obtained from equation (18A).

TABLE 5

VALUES OF C

(Flow from Test-Hole Located below Groundwater Level)

$\frac{L}{r}$	C	$\frac{L}{r}$	C	$\frac{L}{r}$	C
5	19.520	11	28.823	17	37.701
6	21.040	12	30.342	18	39.129
7	22.602	13	31.846	19	40.538
8	24.173	14	33.331	20	41.945
9	25.736	15	34.803	21	43.339
10	27.287	16	36.259	22	44.720

Experimental results are in almost complete agreement with this analysis. Results of electric analogy tests show average values not greater than 12 percent below calculated values for $L/r \geqq 8$. The approximate mathematical method has reasonable validity for $L/r \geqq 5.0$, and almost perfect agreement for $L/r \geqq 20$.

APPENDIX B

FLOW FROM A TEST-HOLE LOCATED ABOVE GROUNDWATER LEVEL

(Development by R. E. Glover)

This development applies when the groundwater table is a considerable distance below the drilled hole used for the field test. The test is made by running water into the hole and noting the depth of water that can be maintained in the hole by a metered flow of water.

Since the gravitational potential must be treated explicitly here, an exact solution would require that an expression be found which would satisfy Laplace's equation within a region possessing radial symmetry with respect to the axis of the hole. At an inner boundary coinciding with the surface of the hole, the pressures would be hydrostatic, while at the outer boundary the pressures would have to be adjusted to zero along some streamline with the gravitational potential accounted for everywhere.

It would be very difficult to find a solution satisfying these requirements, and it will therefore be expedient to use an approximation. This will be obtained through the following procedure. Consider first the case where the surface of the ground is kept supplied with water so that it remains covered to a very small depth. The water will then move downward through the ground under the influence of gravity and the pressure will be zero everywhere. The flow will be at a rate which could be maintained by a unit pressure gradient. Now suppose a point-source of strength q second-feet ($ft.^3$/sec.) is superimposed on the gravitational flow system. This will give rise to pressures and new velocities. At a great distance below the source, the velocities due to the source will be negligibly small and only the velocities due to the gravitational forces will remain. If a cylindrical surface of radius b with its axis vertical and passing through the source is constructed in the bed, all the flow, q, will be confined within it providing b is chosen large enough so that the area πb^2 is sufficient to transmit the flow with the velocities which can be maintained by the gravitational forces. This can be demonstrated by constructing a streamline passing through a point at a distance b from the axis and at a great distance below the source. This streamline will lie on a plane containing the axis and will cut the axis at a distance b/2 above the source. The plot of such a streamline is shown in figure 57. A surface of revolution generated by revolving this streamline about the axis, together with the circular area πb^2, will completely enclose the source. Since there can be no flow across a streamline, it follows that all the flow q must be confined within the surface of revolution. This being the case, it would be possible to replace the surface of revolution with an impermeable membrane and eliminate the flow outside it without interfering with the flow within. Thus a flow pattern for fluid supplied to a bed at a point and flowing through the bed under the action of gravity is obtained. It is now desirable to examine this solution to determine its suitability as the basis of an acceptable approximation.

The solution is a solution of Laplace's equation and the outer boundary is a streamline, so that two of the primary requirements are met; but the outer boundary is not free from pressure and the pressure conditions around a cylindrical boundary

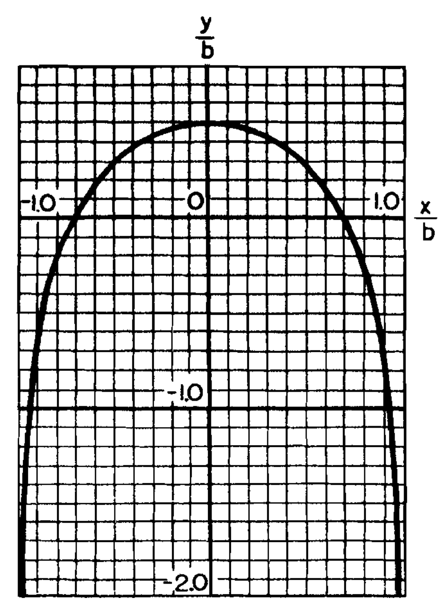

Figure 57 - Gravity Flow Boundary in Unsaturated Material.

representing the surface of the test-hole are not met. The difficulty arising from the pressures at the outer boundary is not regarded as serious, since the pressures are small everywhere and fade away rapidly with increasing distance from the source. The probable net effect is that the actual envelope is slightly outside the surface of revolution near the source. It is considered that the surface of revolution is a sufficiently close approximation to the shape of the actual envelope for the present purposes. The desired pressure distribution along the inner cylindrical boundary can be approximately supplied by using a series of uniformly spaced sources, starting at zero strength at the top of the water surface in the hole and increasing linearly in strength to a maximum at the bottom. A streamline can be found for this combination also which, when rotated about the axis, would generate an outer boundary shaped enough like the actual boundary to be usable for purposes of approximation.

The development of formulas may now proceed. Let

- p = pressure head measured in feet of water,
- K = permeability in feet per second, and
- q = strength of a source, counted positive if the flow is outward.

It should be noted that in a bed where the velocity is proportional to the pressure gradient the flow patterns are superimposable. The source patterns may therefore be superimposed on the gravitational flow, and, since the gravitational flow produces no pressures, the pressures will be due to the sources only. This means that the pressures will be the same as would prevail if the gravitational potential and gravitational flow did not exist. The gravitational flow does, however, change the course of a particle issuing from the source so that instead of traveling radially outward it is given a downward component and its course is effectively confined within the cylinder of radius b. The pressure due to a source of strength q at the point y = h, x = 0, is

$$p = \frac{q}{4\pi K} \cdot \frac{1}{\sqrt{x^2 + (y-h)^2}} \quad \text{. . (1B)}$$

To provide a series of point-sources whose strength increases with depth, let

$$dq = B\ (H - h)\ dh \quad \text{. . . . . (2B)}$$

where H represents the value of y at the water surface in the hole. Then,

$$dp = \frac{B}{4\pi K} \frac{(H-h)}{\sqrt{x^2 + (y-h)^2}}\ dh \quad \text{. . (3B)}$$

and by integration,

$$p = \frac{B}{4\pi K}\left[-\ (H-y)\sinh^{-1}\frac{(y-h)}{x} - \sqrt{x^2 + (y-h)^2}\right]_{h=0}^{h=H} \quad \text{. . . . . (4B)}$$

or

$$p = \frac{B}{4\pi K}\Big[-\ (H-y)\sinh^{-1}\frac{(y-H)}{x} + (H-y)\sinh^{-1}\frac{y}{x} - \sqrt{x^2 + (y-H)^2} + \sqrt{x^2 + y^2}\Big] \quad \text{. . . . . (5B)}$$

Now, if the whole flow is represented by Q,

$$Q = \int_0^H B(H-h)\ dh = -B\left[\frac{(H-h)^2}{2}\right]_0^H$$

$$= B\ \frac{H^2}{2} \quad \text{. . . . . (6B)}$$

Then, by substitution for B,

$$p = \frac{Q}{2\pi H^2 K}\Big[(H-y)\sinh^{-1}\frac{(H-y)}{x} + (H-y)\sinh^{-1}\frac{y}{x} - \sqrt{x^2 + (H-y)^2} + \sqrt{x^2 + y^2}\Big] \quad \text{. . . . . (7B)}$$

At $y = 0$, $x = a$; this is the specification for a point at the boundary of the hole at the bottom. By substitution in equation (7B),

$$p_o = \frac{Q}{2\pi H^2 K}\left[H \sinh^{-1}\left(\frac{H}{a}\right) - \sqrt{a^2 + H^2} + a\right] \quad \ldots\ldots (8B)$$

If $\frac{H}{a}$ is large compared with unity, then approximately,

$$p_o = \frac{QH}{2\pi H^2 K}\left[\sinh^{-1}\left(\frac{H}{a}\right) - 1\right] \quad .. (9B)$$

At this point, $p_o = H$.

The above equation may be solved for K in the form,

$$K = \frac{Q}{2\pi H^2}\left[\sinh^{-1}\left(\frac{H}{a}\right) - 1\right] \quad .. (10B)$$

The radius, b, may be found from the requirement that

$$K\pi b^2 = Q \quad \ldots\ldots\ldots\ldots\ldots (11B)$$

If this substitution is made in equation (10B), the value of b is obtained in the form,

$$b = H\sqrt{\frac{2}{\sinh^{-1}\left(\frac{H}{a}\right) - 1}} \quad .. (12B)$$

It is pointed out again that this derivation is good if the bottom of the drilled hole is an appreciable distance above the original groundwater table and if the radius of the hole is very small compared with the head acting. By this arrangement, water would flow out of the bottom of the well and also the sides of the well.

In order to make the preceding development more useful, equation (10B) may be written in the form,

$$K = \frac{Q}{rH} \cdot \frac{r}{2\pi H}\left[\sinh^{-1}\left(\frac{H}{r}\right) - 1\right] \quad .. (13B)$$

Equation (13B) is of the form,

$$K = \frac{Q}{rH} \cdot \frac{1}{C} \quad \ldots\ldots\ldots\ldots\ldots (14B)$$

where

$$\frac{1}{C} = \frac{r}{2\pi H}\left[\sinh^{-1}\left(\frac{H}{r}\right) - 1\right] \quad \ldots (15B)$$

Equation (15B) may be computed for various values of the ratio H/r within the limits ordinarily employed in the field. Table 6 gives values of C to be used in equation (14B).

TABLE 6

VALUES OF C

(Flow from Test-Hole Located above Groundwater Level)

$\frac{H}{r}$	C	$\frac{H}{r}$	C	$\frac{H}{r}$	C
5.0	23.93	8.5	29.07	14.0	37.70
5.5	24.42	9.0	29.87	15.0	39.24
6.0	25.27	9.5	30.66	16.0	40.75
6.5	26.00	10.0	31.45	17.0	42.27
7.0	26.75	11.0	33.02	18.0	43.77
7.5	27.51	12.0	34.59	19.0	45.25
8.0	28.30	13.0	36.14	20.0	46.71

A comparison of electric analogy test results for conditions similar to those assumed in the above development shows appreciably lower values obtained experimentally. These deviations vary from 25 percent at $H/r = 6$, to 8 percent at $H/r = 20$. This indicates that the approximate mathematical analysis has reasonable validity for $H/r \geqq 10$.

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APPENDIX C

RADIAL FLOW TO A WELL IN AN INFINITE AQUIFER (TIME-DRAWDOWN METHOD)

NOTATION

h = drawdown at observation well
r = distance from center of discharging well to observation well
S = coefficient of storage of aquifer (volume of water a unit drawdown releases from a vertical prism of unit cross-section and depth D)
K = coefficient of permeability
D = depth of aquifer
t = time measured from beginning of discharge
Q = rate of discharge
e = base of Naperian logarithms ($e \cong 2.718$)
C = Euler's constant ($C \cong 0.5772$)
a, s, v, x, y = quantities employed in least square adjustment development.

Consistent units must be used throughout.

The differential equation governing the unsteady radial flow to a well in a confined aquifer is

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{KD}\frac{\partial h}{\partial t} \quad \ldots\ldots (1C)$$

It can be shown [12, 13] that if the boundary conditions are satisfied and the well is pumped at a steady rate, Q, then the drawdown is given by the expression

$$h = \frac{Q}{4\pi KD}\int_{\frac{r^2S}{4KDt}}^{\infty} \frac{e^{-u}}{u}\, du \quad \ldots\ldots (2C)$$

The integral, known as the exponential integral, is a function of the lower limit, and is often abbreviated

$$-Ei\left(-\frac{r^2S}{4KDt}\right)$$

The value of the drawdown can be obtained from the equivalent series:

$$h = \frac{Q}{4\pi KD}\left[-C - \ln\left(\frac{r^2S}{4KDt}\right) + \left(\frac{r^2S}{4KDt}\right) - \frac{1}{2\cdot 2!}\left(\frac{r^2S}{4KDt}\right)^2 + \frac{1}{3\cdot 3!}\left(\frac{r^2S}{4KDt}\right)^3 - \ldots\right] \quad \ldots (3C)$$

After the test has run for a relatively long time and the quantity $\left(\frac{r^2S}{4KDt}\right)$ becomes less than, say 0.02, the series may be approximated by the asymptotic expression:

$$\left.\begin{aligned} h &= -\frac{Q}{4\pi KD}\left[\ln\left(\frac{r^2S}{4KDt}\right) + C\right] \\ \text{or} \\ h &= -\frac{Q}{4\pi KD}\left[\ln\left(\frac{r^2Se^C}{4KDt}\right)\right] \end{aligned}\right] \quad \ldots (4C)$$

but

$e^C \cong 1.781$,

and changing to common logarithms gives

$$h = \frac{0.1832Q}{KD}\log\frac{2.246KDt}{r^2S} \quad \ldots\ldots (5C)$$

[12] Jacob, C. E., "Flow of Ground Water," Chapter V of Engineering Hydraulics by Hunter Rouse, John Wiley and Sons, 1950.

[13] Equations (2C), (3C), (17C), and (18C) follow closely those of W. H. Taylor and E. D. Rainville given in an unpublished memorandum prepared while they were employed as engineers by the U. S. Bureau of Reclamation, Denver, Colorado.

Field data usually involves many drawdown readings at several observation wells at various times. In order to obtain the best values for K and S, it is desirable to adjust such data by least squares in order to arrive at the best drawdown curve for a particular time t. Since h varies logarithmically, the following forms of equation (5C) will be found convenient for least squares adjustment when computing K and S:

$$h = -\frac{0.3665Q}{KD}\left[\log r - 1/2\log\frac{2.246KDt}{S}\right] \ldots\ldots (6C)$$

or

$$h = +\frac{0.1832Q}{KD}\left[\log t - \log\frac{r^2S}{2.246KD}\right] \ldots\ldots\ldots (7C)$$

or

$$h = -\frac{0.1832Q}{KD}\left[\log\frac{r^2}{t} - \log\frac{2.246KD}{S}\right] \ldots\ldots\ldots (8C)$$

Equations (6C), (7C), and (8C) are in the form

$$y(x) = a_o + a_1x \;\ldots\ldots\ldots\ldots (9C)$$

where y corresponds to h and x to log r, log t, and $\log\frac{r^2}{t}$, respectively.

Then, according to Milne[14], having given n pairs of values of x and y, the straight line which fits these data best is determined by values of a_o and a_1 from:

$$s_oa_o + s_1a_1 = v_o \;\ldots\ldots\ldots (10C)$$

$$s_1a_o + s_2a_1 = v_1 \;\ldots\ldots\ldots (11C)$$

The s and v values may be systematically calculated by arranging the recorded data in tabular form and summing the columns as indicated in Table 7.

4 Milne, W. E., Numerical Calculus, Princeton University Press, 1949, pp. 242 - 245.

TABLE 7

CALCULATION OF s AND v VALUES

Observation Number	x^o	x	x^2	y	xy
1	1	x_1	x_1^2	y_1	x_1y_1
2	1	x_2	x_2^2	y_2	x_2y_2
...	...	...	...	...	...
n	1	x_n	x_n^2	y_n	x_ny_n
$\sum$	s_o	s_1	s_2	v_o	v_1

An example will demonstrate the procedure to emplov when using field data. On the Oahe Unit of the Missouri River Basin, tests have been made to determine K, the coefficient of permeability, and S, the storage coefficient of an artesian aquifer underlying certain agricultural lands. The data will be used in equation (6C), since drawdown observations were made simultaneously at observation wells. The results are tabulated in Table 8. For Pump Test No. 9 the following data were recorded:

t = 1,224,000 seconds
Q = 300 gal. per minute = 0.668 $ft.^3$/sec.
D = 152 feet

Inserting values of s_o, s_1, s_2, v_o, and v_1, from Table 8, into equations (10C) and (11C) gives

$$8a_o + 19.867a_1 = 47.42 \;\ldots\ldots (12C)$$

$$19.867a_o + 50.658a_1 = 112.078 \;\ldots (13C)$$

and solving, we find

$$a_o = 16.614 \text{ and } a_1 = -4.3032.$$

From equation (9C), we may write

$$h = 16.614 - 4.3032 \log r \;\ldots\ldots (14C)$$

and by comparing the coefficients of log r in this and equation (6C), it is seen that

$$K = -\frac{0.3665Q}{a_1D} = \frac{0.3665Q}{4.3032D} \;\ldots\ldots (15C)$$

Since Q = 0.668 cfs and D = 152 feet, the coefficient of permeability, K, may be determined directly from (15C). This gives

K = 0.000374 ft. per sec.

Again, by equating the constant terms of (14C) and (6C), it is seen that

$$S = - \frac{0.8230Qt}{a_1 \text{ antilog } (- \frac{2a_o}{a_1})} \quad \ldots\ldots (16C)$$

The storage coefficient, S, may be obtained from (16C) by noting that:

$$\text{antilog } (- \frac{2a_o}{a_1}) = \text{antilog } 7.7217$$

$$= 5.269 \times 10^7$$

so that

$$S = \frac{0.8230 \times 0.668 \times 1{,}224{,}000}{4.3032 \times 5.269 \times 10^7}$$

$$= 0.00297.$$

Having computed K and S, a check should be made to insure that all observations fall within the restriction $(\frac{r^2S}{4KDt}) < 0.02$. If any one of the data is without this limitation, it should be discarded, and K and S recomputed. In this instance, for observation well W10, it was found $(\frac{r^2S}{4KDt}) = 0.03$. Omitting this datum, the revised values of K and S may be determined as

K = 0.000349 ft. per sec.
S = 0.00458

The method outlined above for the determination of K and S is excellent for resolving field data in an office. Often, however, the engineer or geologist may wish to analyze his data in part in the field. In this case the following equation will be useful for determining the coefficient of permeability.

When two observations are made at the same time in adjacent wells, K is given approximately by the equation

$$K = \frac{Q}{4\pi D} \cdot \frac{\ln (\frac{r_2}{r_1})^2}{h_{(r_1, t)} - h_{(r_2, t)}} \quad \ldots (17C)$$

$h_{(r,t)}$ is the observed drawdown at time t in a well at distance r from the test well. Equation (17C) is valid where the ratio of

$$\left| \frac{(r_2^2 - r_1^2)S}{4KDt} \right| \text{ to } \left| \ln (\frac{r_2}{r_1})^2 \right|$$

is small, say less than 0.02.

TABLE 8

CALCULATION OF s AND v (EXAMPLE)

Observation Well	r	x^o	$x = \log r$	$x^2 = (\log r)^2$	$y = h$	$xy = h \log r$
S2	96	1	1.982	3.928	8.14	16.133
W2	98	1	1.991	3.964	8.09	16.107
S4	189	1	2.276	5.180	6.66	15.158
W4	199	1	2.299	5.285	6.90	15.863
S6	390	1	2.591	6.713	5.24	13.577
W6	400	1	2.602	6.770	5.57	14.493
S8	790	1	2.898	8.398	3.84	11.128
W10	1692	1	3.228	10.420	2.98	9.619
		s_o	s_1	s_2	v_o	v_1
		8	19.867	50.658	47.42	112.078

When two observations are made at different times in the same well, K is given approximately by the equation

$$K = \frac{Q}{4\pi D} \cdot \frac{\ln\left(\frac{t_2}{t_1}\right)}{h_{(r,\, t_2)} - h_{(r,\, t_1)}} \quad \ldots (18C)$$

Equation (18C) is valid where the ratio

$$\left| \frac{Sr^2}{4KDt_2} - \frac{Sr^2}{4KDt_1} \right| \text{ to } \left| \ln\left(\frac{t_2}{t_1}\right) \right|$$

is small, say less than 0.02. Equation (18C) may also be used for determining K by noting the recovery of water level in a well after a period of pumping followed by a period of stoppage. In this case t_2 is the total time from the beginning of pumping to the time of recovery observation, and t_1 is the time since pumping stopped to the time of observation.

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